EXISTENCE OF MINIMIZERS FOR NONCONVEX, NONCOERCIVE SIMPLE INTEGRALS

P. CELADA – S. PERROTTA

Abstract. We consider the problem of minimizing autonomous, simple integrals like

\[(P) \quad \min \left\{ \int_0^T f(x(t), x'(t)) \, dt : \ x \in AC([0, T]) \text{ with } x(0) = x_0 \text{ and } x(T) = x_T \right\}\]

where \( f: \mathbb{R} \times \mathbb{R} \to [0, \infty] \) is a possibly nonconvex function with either superlinear or slow growth at infinity. Assuming that the relaxed problem \((P^{**})\) obtained from \((P)\) by replacing \( f \) with its convex envelope \( f^{**} \) with respect to the derivative variable \( x' \) admits a solution, we prove attainment for \((P)\) under mild regularity and growth assumptions on \( f \) and \( f^{**} \). We discuss various instances of growth conditions on \( f \) that yield solutions to the corresponding relaxed problem \((P^{**})\) and we present examples that show that the hypotheses on \( f \) and \( f^{**} \) considered here for attainment are essentially sharp.

0. Introduction

This paper deals with the existence of solutions to variational problems for autonomous, simple integrals like

\[(P) \quad \min \left\{ \int_0^T f(x(t), x'(t)) \, dt : \ x \in AC([0, T]) \text{ with } x(0) = x_0 \text{ and } x(T) = x_T \right\}\]

where the Lagrangean function \( f: \mathbb{R} \times \mathbb{R} \to [0, \infty] \) is a possibly nonconvex function of its second argument \( x' \). Though the emphasis here is on the lack of convexity of \( f \), we remark that we wish to consider either problems with slow growth, i.e. \( f(\eta, \xi) \) has no superlinear growth as \( |\xi| \to \infty \), and problems with an extended-valued Lagrangean \( f \) as it happens in the case of one-sided constraints on the derivative like \( x' \geq 0 \) or \( x' > 0 \) almost everywhere on \([0, T]\).

As is well known, the lack of convexity of \( f(\eta, \xi) \) with respect to \( \xi \) affects the sequential lower semicontinuity of the integral with respect to weak convergence in \( AC([0, T]) \), thus ruling out the possibility of establishing the existence of optimal configurations by means of the direct method of the Calculus of Variations. Yet, attainment is a quite typical behaviour for variational, simple integrals and the basic question for nonconvex minimum problems like \((P)\) becomes that of finding which conditions other than convexity of \( f(\eta, \xi) \) with respect to \( \xi \) yield solutions to \((P)\).

This question has been widely investigated in recent years, mainly when \( f \) has a special form like \( f(\eta, \xi) = h(\xi) + g(\eta) \) or \( f(\eta, \xi) = g(\eta)h(\xi) \) with nonnegative \( g \) and \( h \). In either cases, a fairly complete understanding of attainment versus nonattainment phenomena is now available: roughly speaking, attainment occurs provided \( g \in C(\mathbb{R}) \) is such that (i) every point \( t \in \mathbb{R} \) lies between two intervals where \( g \) is monotone, i.e. \( g \) does not oscillate too fast, and (ii) \( g \) has no strict, local minima. Moreover, well known Bolza’s type examples like

\[\min \left\{ \int_0^T \left[ \left( |x'(t)|^2 - 1 \right)^2 + |x(t)|^2 \right] \, dt : \ x \in AC([0, T]) \text{ with } x(0) = x(T) = 0 \right\}\]
and
\[
\min \left\{ \int_0^T \left( 1 + |x(t)|^2 \right) \left[ 1 + \left( |x'(t)|^2 - 1 \right)^2 \right] \; dt : \; x \in AC([0, T]) \text{ with } x(0) = x(T) = 0 \right\}
\]

show that attainment is not to be expected to hold in general if the latter assumption on \( g \) is dropped, unless \( h \) is supposed to be convex at zero, i.e. the values at zero of \( h \) and its convex envelope \( h^{**} \) coincide (see [12] and [17] for a complete discussion of this issue). Among the many related papers, we refer to [1], [18], [4], [5], [6], [16], [8] and [9] for sum-like integrals and to [14], [15] and [2] for product-like integrals. We mention also the already quoted papers [12] and [17] for a somewhat different point of view on the subject.

As regards the case of nonconvex Lagrangean functions \( f \) of general form, we mention [19], [14], [13], [15] and [20]. Roughly speaking, in these papers, assuming that either \( f \) and its convex envelope \( f^{**} \) with respect to \( \xi \) are smooth, attainment for \((\mathcal{P})\) is proved when the continuous function

\[
f^{**}(\eta, \xi) - \xi \frac{\partial f^{**}}{\partial \xi}(\eta, \xi)
\]

is either monotone or concave as a function of \( \eta \) for every \( \xi \) or possibly on the sections with fixed \( \xi \) of the set \( \{f^{**} < f\} \) only (see [20]). Note that, letting \( f^* \) be the polar function of \( f(\eta, \xi) \) with respect to \( \xi \), the function above coincides with

\[
-f^* \left( \eta, \frac{\partial f^{**}}{\partial \xi}(\eta, \xi) \right),
\]

i.e. the value at the origin of the supporting affine function to the graph of \( \xi \rightarrow f^{**}(\eta, \xi) \) through the point \((\eta, \xi)\). Hence, according to these papers, attainment for \((\mathcal{P})\) seems to require a very special, global (or possibly local as in [20]) behaviour of either (0.1) or (0.2) as a function of \( \eta \) for every \( \xi \) like monotonicity or concavity. However, in the special cases of variational problems \((\mathcal{P})\) featuring smooth, sum-like or product-like Lagrangean functions \( f \), (0.1) and (0.2) turn in

\[
\left\{ \begin{array}{l}
\left[ h^{**}(\xi) - \xi (h^{**})'(\xi) \right] + g(\eta) = -h^* \left( (h^{**})'(\xi) \right) + g(\eta), \\
g(\eta) \left[ h^{**}(\xi) - \xi (h^{**})'(\xi) \right] = g(\eta) \left[ -h^* \left( (h^{**})'(\xi) \right) \right],
\end{array} \right.
\]

respectively. Hence, the monotonicity or concavity assumptions on (0.1) and (0.2) as functions of \( \eta \) reduce to the requirement that \( g \) share the same property on the whole real line in case of sum-like integrals and that \( g \) be monotone or possibly convex provided \( h^* \left( (h^{**})'(\xi) \right) \geq 0 \) for every \( \xi \) in the product-like case. By contrast, the existence results mentioned before for these special problems call only for weaker properties of \( g \), namely no oscillations on smaller and smaller scale and no strict local minima.

Thus, there is a gap between the available attainment results for sum- or product-like, nonconvex, variational problems on one hand and the same problems with a Lagrangean \( f \) of general form on other hand and the aim of this paper is precisely to fill this gap. Indeed, we are going to show that the hypotheses on \( f^{**} \) that yield attainment for \((\mathcal{P})\) in the general case look even weaker than they appear in the case of sum-like or product-like integrals.

To this aim, provided \( f \) enjoys mild regularity and growth assumptions (see Theorem 1.2), we associate with the convex envelope \( f^{**} \) of \( f \) with respect to \( \xi \) a function \( Ef^{**} : \mathbb{R} \times \mathbb{R} \rightarrow [-\infty, \infty] \) whose value at a point \((\eta, \xi)\) is, roughly speaking, the value at the origin of the supporting affine function to the graph of \( \xi \rightarrow f^{**}(\eta, \xi) \) through the point \((\eta, \xi)\) and which reduces to (0.1) and (0.2) for smooth convex envelopes \( f^{**} \). Then, assuming also that the relaxed problem \((\mathcal{P}^{**})\) obtained
from \((P)\) by replacing \(f\) with its convex envelope \(f^{**}\), i.e.
\[
\min \left\{ \int_0^T f^{**}(x(t), x'(t)) \, dt : x \in AC([0, T]) \text{ with } x(0) = x_0 \text{ and } x(T) = x_T \right\},
\]
admits a solution, we prove attainment for \((P)\) provided \(Ef^{**}\) and \(f^{**}\) have the following qualitative, local behaviour on the set \(\{f^{**} < f\}\):
\[(a) \text{ if } f(\eta_0, \xi_0) < f^{**}(\eta_0, \xi_0), \text{ there is } \delta = \delta(\eta_0, \xi_0) > 0 \text{ such that } \eta \to Ef^{**}(\eta, \xi_0) \text{ is monotone on both intervals } [\eta_0 - \delta, \eta_0] \text{ and } [\eta_0, \eta_0 + \delta];
\]
and, whenever the section of \(\{f^{**} < f\}\) with \(\xi = 0\) is not empty,
\[(b) \text{ the function } \eta \to f^{**}(\eta, 0) \text{ has no strict, local minima on such section.}
\]
We wish to point out that, in the Bolza’s type examples mentioned above, the set \(\{f^{**} < f\}\) is given in either cases by \(\mathbb{R} \times (-1, 1)\), that \(f^{**}(\eta, 0)\) is given by \(\eta^2\) and \(1 + \eta^2\) respectively and that all the other assumptions of our result are satisfied. Thus, nonattainment for those problems is a direct consequence of the failure of (b).
We refer to Section 1 ahead for the exact statement of our result, for a more detailed discussion of its hypotheses and for some examples.
Finally, we wish to remark that the existence result for the nonconvex problem \((P)\) we are going to prove is based on the assumption of attainment for the corresponding relaxed problem \((P^{**})\) and thereby can be applied to nonconvex problems featuring either superlinear or slow growth at infinity provided the associated relaxed problem admits a solution. Indeed, besides the standard case of functions \(f\) having superlinear growth at infinity (see Corollary 1.3) for which the existence of solutions for the corresponding relaxed problem \((P^{**})\) follows immediately from the direct method of the Calculus of Variations, we consider also the case of functions \(f\) with slow growth at infinity (see Corollary 1.4) for which attainment for the relaxed problem \((P^{**})\) can be obtained by applying the existence result of [7].
The remaining part of the paper is organized as follows. In the next section, we introduce some notations, we recall some well known preliminary results and we state the main result (Theorem 1.2) and prove its consequences (Corollaries 1.3 and 1.4). Then, in the following Section 2, we prove some technical results that will be needed in the proof of Theorem 1.2 presented in the last Section 3.

1. Notations and statement of the main results

We begin by recalling some elementary definitions, notations and results, mostly from convex analysis and measure theory.
If \(A \subset \mathbb{R}^n\), we let \(\text{int} (A), \overline{A}\) and \(\partial A\) be the interior, the closure and the boundary of \(A\) respectively. The effective domain of a function \(g: A \to (-\infty, \infty]\) is the subset of \(A\) defined by
\[
\text{dom} (g) = \{ \xi \in A : g(\xi) < \infty \}
\]
and \(g\) itself is said to be proper whenever its effective domain is not empty. Now, let \(g: \mathbb{R} \to [0, \infty]\) be a proper, lower semicontinuous function. We recall that \(g\) is said to be subdifferentiable at a point \(\xi \in \text{dom} (g)\) if there exists \(d \in \mathbb{R}\) such that
\[
(1.1) \quad g(\zeta) \geq g(\xi) + d(\zeta - \xi), \quad \zeta \in \mathbb{R}.
\]
Every such \(d\) is a subgradient of \(g\) at \(\xi\) and the set of all such numbers \(d\) is the subdifferential \(\partial g(\xi)\) of \(g\) at \(\xi\). When \(g\) is also convex, \(\partial g(\xi)\) is a nonempty, compact interval for every \(\xi \in \text{int} (\text{dom} (g))\) and \(g\) turns out to be locally Lipschitz continuous on \(\text{int} (\text{dom} (g))\) so that \(\partial g(\xi) = \{ g'(\xi) \} \) for almost every \(\xi \in \text{int} (\text{dom} (g))\).
We also denote the space of all smooth, compactly supported functions on the real line by \( \mathcal{D} \) with respect to the Sobolev norm \( \mathcal{D}_p \) the space of absolutely continuous functions on \([0, T]\) for the Lebesgue space of integrable functions on \([0, T]\).

It is plain that, for every such point \( x \in E \) that Lebesgue’s differentiation theorem states that almost every point of the proper, lower semicontinuous, convex function \( g \)

We recall also that, if \( g : \mathbb{R} \to [0, \infty) \) is proper and lower semicontinuous, the \emph{polar function} of \( g \) is the proper, lower semicontinuous, convex function \( g^* : \mathbb{R} \to (-\infty, \infty] \) defined by

\[
g^*(\zeta) = \sup \{ \xi \zeta - g(\xi) : \xi \in \mathbb{R} \}, \quad \zeta \in \mathbb{R},
\]

(see [11]) and that the \emph{bipolar function} or \emph{convex envelope} of \( g \) is the polar \( g^{**} : \mathbb{R} \to [0, \infty] \) of \( g^* \). Thus, \( g^{**} \) is a proper, lower semicontinuous, convex function such that

\[
(1.2) \quad g^{**}(\xi) \leq g(\xi) \text{ for every } \xi \in \mathbb{R};
\]

\[
(1.3) \quad g^{**}(\xi) = g(\xi) \text{ for every } \xi \in \mathbb{R} \setminus \text{int} (\text{dom } (g^{**}));
\]

\[
(1.4) \quad \text{the set } \{g^{**} < g\} \text{ is open};
\]

\[
(1.5) \quad g^{**} \text{ is affine on the connected components of } \{g^{**} < g\};
\]

\[
(1.6) \quad \text{the closure of each connected component of } \{g^{**} < g\} \text{ is contained in } \text{dom } (g^{**}).
\]

Moreover, we recall that, whenever \( d \in \mathbb{R} \) is a subgradient of \( g^{**} \) at some point \( \xi \in \text{dom } (g^{**}) \), the values of \( g^{**}(\xi) \) and \( g^*(d) \) are related by

\[
(1.7) \quad g^{**}(\xi) + g^*(d) = d\xi
\]

(see [11]) because of the equality \( g^{***} = g^* \). Hence, writing (1.1) with \( g^{**} \) instead of \( g \), it follows that the value at the origin of the supporting affine function to the graph of \( g^{**} \) through the point \((\xi, g^{**}(\xi))\) with slope \( d \) is given by \(-g^*(d)\).

As to measure theoretic notations and results, we denote the Lebesgue measure of a measurable subset \( E \) of \( \mathbb{R} \) by \( |E| \) and we recall that a family of nondegenerate, compact intervals \( \mathcal{K} \) is said to \emph{shrink} at some point \( x \in \mathbb{R} \) if \( x \in K \) for every \( K \in \mathcal{K} \) and \( \inf \{ |K| : K \in \mathcal{K} \} = 0 \). We recall also that a \emph{Vitali covering} of a measurable set \( E \) is a family of nondegenerate, compact intervals \( \mathcal{K} \) such that, for a.e. \( x \in E \), the subfamily of those intervals \( K \in \mathcal{K} \) containing \( x \) shrinks at \( x \) itself.

We emphasize that, in the definitions above, the intervals \( K \) associated with \( x \) need not be neither centered at \( x \) nor nested. Then, Vitali’s covering theorem states that every such covering contains a (at most) countable subfamily of sets \( \{K_n\}_n \) consisting of pairwise disjoint intervals that cover \( E \) up to a negligible set, i.e. \( |E \setminus (\cup_n K_n)| = 0 \). We also recall that a point \( x \in \mathbb{R} \) is said to be a \emph{density point} for a measurable set \( E \) if

\[
\frac{1}{2\varepsilon} |(x - \varepsilon, x + \varepsilon) \cap E| \to 1 \quad \text{or equivalently} \quad \frac{1}{2\varepsilon} |(x - \varepsilon, x + \varepsilon) \setminus E| \to 0 \quad \text{as } \varepsilon \to 0_+
\]

and that Lebesgue’s differentiation theorem states that almost every point of \( E \) is a density point. It is plain that, for every such point \( x \),

\[
(1.8) \quad \frac{|K \setminus E|}{|K|} \to 0 \quad \text{as } K \in \mathcal{K} \text{ and } |K| \to 0
\]

whenever \( \mathcal{K} \) shrinks at \( x \).

As regards functional theoretic notations, we let \( T \) be a positive number, we use standard notations for the Lebesgue space of integrable functions on \([0, T]\) and its norm and we write \( AC([0, T]) \) for the space of absolutely continuous functions on \([0, T]\) which turns out to be a Banach space with respect to the Sobolev norm

\[
\|x\|_{1,1} = \int_0^T \left[ |x(t)| + |x'(t)| \right] dt, \quad x \in AC([0, T]).
\]

We also denote the space of all smooth, compactly supported functions on the real line by \( \mathcal{D}(\mathbb{R}) \).
Now, we introduce the class of functionals we are going to consider in the sequel. Given a proper and lower semicontinuous function \( f: \mathbb{R} \times \mathbb{R} \to [0, \infty] \) we consider the following integral functional
\[
I(x) = \int_0^T f(x(t), x'(t)) \, dt, \quad x \in AC([0, T]),
\]
and the associated minimum problem
\[
(\mathcal{P}) \quad \min \{ I(x): \ x \in AC([0, T]) \text{ with } x(0) = x_0 \text{ and } x(T) = x_T \}
\]
with \( x_0, x_T \in \mathbb{R} \). We denote the polar and the bipolar functions of \( f \) with respect to the second variable \( \xi \) by \( f^*: \mathbb{R} \times \mathbb{R} \to (-\infty, \infty] \) and \( f^{**}: \mathbb{R} \times \mathbb{R} \to [0, \infty] \) respectively and, for every \( \eta \in \mathbb{R} \), we denote also the subdifferential of \( \xi \to f^{**}(\eta, \xi) \) at the point \( \xi \in \mathbb{R} \) by \( \partial f^{**}(\eta, \xi) \). Then, we consider the functional
\[
I^{**}(x) = \int_0^T f^{**}(x(t), x'(t)) \, dt, \quad x \in AC([0, T]),
\]
and the associated minimum problem
\[
(\mathcal{P}^{**}) \quad \min \{ I^{**}(x): \ x \in AC([0, T]) \text{ with } x(0) = x_0 \text{ and } x(T) = x_T \}
\]
which we loosely refer to as the relaxed functional and the relaxed minimum problem respectively. It is plain that \( I^{**} \leq I \) on \( AC([0, T]) \) so that any solution \( x \) to \( (\mathcal{P}^{**}) \) satisfying \( f^{**}(x, x') = f(x, x') \) almost everywhere on \([0, T]\) is a solution to \( (\mathcal{P}) \) as well. Moreover, \( I^{**} \) is sequentially weakly lower semicontinuous on the set of competing functions \( \{ x \in AC([0, T]): \ x(0) = x_0 \text{ and } x(T) = x_T \} \).

In the sequel, we shall consider the following assumptions on the functions \( f \) and \( f^{**}: \)

(H1) \( \text{dom}(f) = \text{dom}(f^{**}) = \mathbb{R} \times C \) where \( C \) is a nondegenerate interval;

(H2) \( f \) and \( f^{**} \) are continuous on \( \mathbb{R} \times \text{int}(C) \).

If the function \( f \) satisfies (H1), it follows that \( f^{**}(\eta, \xi) = f(\eta, \xi) \) for every \( \eta \in \mathbb{R} \) and \( \xi \in \mathbb{R} \setminus \text{int}(C) \) because of (1.3) and moreover, if \( f \) satisfies also (H2), the detachment set \( D \) defined by
\[
D = \{ (\eta, \xi) \in \mathbb{R} \times \mathbb{R}: \ f^{**}(\eta, \xi) < f(\eta, \xi) \}
\]
is an open subset of \( \mathbb{R} \times \text{int}(C) \). In the sequel, we shall denote the sections of the detachment set \( D \) with \( \eta \) and \( \xi \) fixed by \( D_\eta = \{ \xi \in \mathbb{R}: \ (\eta, \xi) \in D \} \) and \( D_\xi = \{ \eta \in \mathbb{R}: \ (\eta, \xi) \in D \} \) respectively.

Now, consider the mapping \( Ef^{**}: \mathbb{R} \times \mathbb{R} \to [-\infty, \infty] \) defined by
\[
Ef^{**}(\eta, \xi) = \sup \{ -f^*(\eta, d): \ d \in \partial f^{**}(\eta, \xi) \}, \quad (\eta, \xi) \in \mathbb{R} \times \mathbb{R},
\]
where, as usual, the supremum of the empty set is set equal to \(-\infty\). Note that, if it happens that \( f^{**} \) is smooth, say \( f^{**} \in C^1(\mathbb{R} \times \mathbb{R}) \), then \( Ef^{**} \) reduces to the continuous function already considered in (0.1), i.e.
\[
Ef^{**}(\eta, \xi) = f^{**}(\eta, \xi) - \xi \frac{\partial f^{**}}{\partial \xi}(\eta, \xi), \quad (\eta, \xi) \in \mathbb{R} \times \mathbb{R},
\]
because of the basic equality (1.7).

On the function \( f \), we shall consider also the following growth assumption

(H3) \( \lim_{|\xi| \to \infty} \sup \{ Ef^{**}(\eta, \xi): \ |\eta| \leq R \} = -\infty \) for every \( R \geq 0 \).

Note that all functions \( f \) satisfying (H2) and (H3) have the following property: for every positive number \( R \), there exist two numbers \( \alpha > 0 \) and \( \beta \geq 0 \) depending on \( R \) such that
\[
f^{**}(\eta, \xi) \geq \alpha |\xi| - \beta, \quad \text{for every } |\eta| \leq R \text{ and every } \xi \in \mathbb{R}.
\]
The growth condition (H3) is strictly weaker than superlinearity at infinity. Indeed, it is easy to see that, if a proper and lower semicontinuous function \( f: \mathbb{R} \times \mathbb{R} \to [0, \infty] \) satisfies (H1), (H2) and has the further property that, for every given \( R \geq 0 \), \( f(\eta, \xi) \geq \theta(|\xi|) \) for every \( |\eta| \leq R \) and \( \xi \in \mathbb{R} \)
for some suitable function \(\theta: [0, \infty) \to \mathbb{R}\) depending on \(R\) such that \(\theta(|\xi|) / |\xi| \to \infty\) as \(|\xi| \to \infty\), then also (H3) is satisfied (see [7] for instance). By contrast, the function
\[
f(\eta, \xi) = f(\xi) = |\xi| - \log(1 + |\xi|), \quad \xi \in \mathbb{R},
\]
provides a simple example of a convex function satisfying (H3) and having linear growth at infinity. We refer to [3] for interesting results on the relationship between (H3) and the regularity of solutions to \((P^{**})\).

The properties of the restriction of the function \(Ef^{**}\) to the detachment set \(D\) are gathered in the following proposition.

Proposition 1.1. Let \(f: \mathbb{R} \times \mathbb{R} \to [0, \infty]\) be a proper and lower semicontinuous function satisfying (H1) and (H2) and let \(D\) be the detachment set defined by (1.9). Then,

(a) there exists \(d: D \to \mathbb{R}\) such that \(\partial f^{**}(\eta, \xi) = \{d(\eta, \xi)\}\) for every \((\eta, \xi) \in D\);
(b) \(Ef^{**}(\eta, \xi) = -f^*(\eta, d(\eta, \xi))\) for every \((\eta, \xi) \in D\);
(c) \(Ef^{**}\) is finite-valued on \(D\) and both \(Ef^{**}\) and \(d\) are continuous on \(D\);
(d) the restrictions \(\xi \in D_\eta \to Ef^{**}(\eta, \xi)\) and \(\xi \in D_\eta \to d(\eta, \xi)\) are constant on the connected components of \(D_\eta\);
(e) if \(D^0 \neq \emptyset\), then \(Ef^{**}(\eta, 0) = f^*(\eta, 0)\) for every \(\eta \in D^0\).

Proof. For every nonempty section \(D_\eta\) the function \(\xi \in \mathbb{R} \to f^*(\eta, \xi)\) is affine on the connected components of \(D_\eta\) because of (1.5). Hence, it is differentiable at every point \(\xi\) of \(D_\eta\) so that (a) holds with \(d(\eta, \xi)\) given by the partial derivative of \(f^*\) with respect to \(\xi\) at the point \((\eta, \xi)\) and (b) follows from (a) and the definition of \(Ef^{**}\). Then recall that
\[
f^*(\eta, d(\eta, \xi)) = \xi d(\eta, \xi) - f^*(\eta, \xi), \quad (\eta, \xi) \in D,
\]
because of (1.7). The right hand side of this equality is finite since \(D \subset \mathbb{R} \times \text{int}(C) \subset \text{dom}(f^*)\) and moreover \(f^*\) is continuous on \(D\) because of (H2). As to \(d\), its restriction \(\xi \in D_\eta \to d(\eta, \xi)\) is constant on the connected components of \(D_\eta\) so that, for every rectangle \(Q = [\eta_1, \eta_2] \times [\xi_1, \xi_2]\) contained in the detachment set \(D\), we have
\[
d(\eta, \xi) = \frac{f^*(\eta, \xi_1) - f^*(\eta, \xi_2)}{\xi_2 - \xi_1}, \quad (\eta, \xi) \in Q.
\]
Thus, \(d\) too is continuous on \(D\) and (c) and (d) follow from (b), (1.11) and (1.12). Finally, (e) follows immediately from (a) and (1.11). \(\square\)

After these preliminaries, we can state the main result of the paper.

Theorem 1.2. Let \(f: \mathbb{R} \times \mathbb{R} \to [0, \infty]\) be a proper and lower semicontinuous function satisfying (H1), (H2) and (H3). Assume also that the following properties hold:

1. (1.13) for every \((\eta_0, \xi_0) \in D\), there is \(\delta = \delta(\eta_0, \xi_0) > 0\) such that \([\eta_0 - \delta, \eta_0 + \delta] \subset D^0\) and such that the restriction \(\eta \in [\eta_0 - \delta, \eta_0 + \delta] \to Ef^{**}(\eta, \xi_0)\) is monotone on each interval \([\eta_0 - \delta, \eta_0]\) and \([\eta_0, \eta_0 + \delta]\);
2. (1.14) if \(D^0 \neq \emptyset\), the restriction \(\eta \in D^0 \to f^*(\eta, 0)\) has no strict, local minima on \(D^0\).

Then, if the relaxed problem \((P^{**})\) has a solution, the nonconvex problem \((P)\) has a solution too.

As already pointed out in the Introduction, the hypothesis (1.14) cannot be dropped without affecting attainment for \((P)\).

Then, we complete the previous result by presenting two instances of growth hypotheses on \(f\) ensuring the existence of solutions to the relaxed problem \((P^{**})\) and hence to the nonconvex problem \((P)\) by Theorem 1.2. The first one is the familiar case of functions \(f\) having superlinear
growth at infinity whereas the second, a simple application of the existence result of [7], applies to problems featuring Lagrangean functions $f$ with slow growth at infinity. We wish to remark that both results apply to nonconvex problems featuring one-sided constraints on the derivative like $x' \geq 0$ or $x' > 0$ a.e. on $[0, T]$.

**Corollary 1.3.** Let $f: \mathbb{R} \times \mathbb{R} \to [0, \infty]$ be a proper and lower semicontinuous function such that all the hypotheses of Theorem 1.2 hold with (H3) replaced by the following:

\begin{align}
(1.15) \ f(\eta, \xi) & \geq \alpha |\xi| - \beta, (\eta, \xi) \in \mathbb{R} \times \mathbb{R}, \text{ for some } \alpha > 0 \text{ and } \beta \geq 0; \\
(1.16) \ \text{for every } R \geq 0, \text{ there exists } \theta: [0, \infty) \to \mathbb{R} \text{ such that } f(\eta, \xi) \geq \theta(|\xi|) \text{ for every } |\eta| \leq R \text{ and } \xi \in \mathbb{R} \text{ and } \theta(|\xi|)/|\xi| \to \infty \text{ as } |\xi| \to \infty. 
\end{align}

Then, the nonconvex problem $(P)$ admits (at least) a solution for every boundary data $x_0, x_T \in \mathbb{R}$.

**Proof.** First, recall that (1.16) implies (H3) so that all the hypotheses of Theorem 1.2 hold. Then, assume there is some feasible function $\varphi \in AC([0, T])$ such that $I^{**}(\varphi) = c < \infty$ otherwise there is nothing to prove and set $A = \{x \in AC([0, T]): x(0) = x_0, x(T) = x_T \text{ and } I^{**}(x) \leq c\}$. All functions $x \in A$ are uniformly bounded because of (1.15). Hence, $I^{**}$ is coercive on $A$ by (1.16) and lower semicontinuous on the same set with respect to weak convergence in $AC([0, T])$ (see Theorem 2.1, Chapter 8 in [11] for instance). Thus, $(P^{**})$ admits a solution and the conclusion follows from Theorem 1.2. 

**Corollary 1.4.** Let $f: \mathbb{R} \times \mathbb{R} \to [0, \infty]$ be a proper and lower semicontinuous function such that all the hypotheses of Theorem 1.2 hold. Assume also that

\begin{align}
(1.17) \ C \text{ is a cone;} \\
(1.18) \ \partial f^{**}(\eta, \xi) \neq \emptyset \text{ for every } (\eta, \xi) \in \mathbb{R} \times C; \\
(1.19) \ f(\eta, \xi) \geq \alpha |\xi| - \beta, (\eta, \xi) \in \mathbb{R} \times \mathbb{R}, \text{ for some } \alpha > 0 \text{ and } \beta \geq 0.
\end{align}

Then, the nonconvex problem $(P)$ admits (at least) a solution for every boundary data $x_0, x_T \in \mathbb{R}$.

Recall that a cone in $\mathbb{R}$ is either $\mathbb{R}$ itself or any open or closed half line starting at zero. Note that (1.18) is automatically fulfilled if $C$ is open and recall also that the detachment set $D$ is contained in $\mathbb{R} \times \text{int } (C)$. Hence, unless $C$ is the whole real line, the section $D^0 \times \mathbb{R}$ with $\xi = 0$ is empty so that (1.14) is automatically fulfilled too.

**Proof of Corollary 1.4.** The very same computations of [2], Corollary 1.4 show that the hypotheses of the existence result of [7] hold for the relaxed problem $(P^{**})$. Thus, $(P^{**})$ admits a solution and the conclusion follows from Theorem 1.2. 

Finally, we end this section by presenting two examples of nonconvex problems which the previous results apply to. They are not meant to be meaningful from the point of view of applications. We just want to illustrate the scope of application of the previous results by showing examples of problems which the previously known attainment results do not apply.

**Example 1.5.** Let $f: \mathbb{R} \times \mathbb{R} \to [0, \infty)$ be defined by

$$f(\eta, \xi) = |\xi - a(\eta)|^2 |\xi - b(\eta)|^2 + c(\eta), \quad (\eta, \xi) \in \mathbb{R} \times \mathbb{R},$$

where the coefficients $a, b, c \in \mathcal{C}(\mathbb{R})$ are such that

\begin{align}
(1.20) \ c(\eta) \geq \alpha \max \{|a(\eta)|, |b(\eta)|\}, \quad \eta \in \mathbb{R},
\end{align}

for some $\alpha > 0$. Note that, if $a$ and $b$ are bounded, the condition above can always be satisfied by any bounded below function $c$ by possibly adding a positive constant to $c$ itself. Here, we obviously
assume that \( a(\eta_0) \neq b(\eta_0) \) for some \( \eta_0 \) otherwise \( f \) would be convex with respect to \( \xi \) and we can also assume without loss of generality that \( a(\eta) \leq b(\eta) \) for every \( \eta \in \mathbb{R} \).

Then, the convex envelope \( f^{**} \) of \( f \) with respect to \( \xi \) is given by

\[
f^{**}(\eta, \xi) = \begin{cases} 
|\xi - a(\eta)|^2 \xi - b(\eta)|^2 + c(\eta), & \xi \leq a(\eta) \text{ or } \xi \geq b(\eta), \\
\xi, & a(\eta) \leq \xi \leq b(\eta), 
\end{cases}
\]

and the detachment set \( D \) by \( D = \{(\eta, \xi) : a(\eta) < \xi < b(\eta)\} \).

It is clear that (H1) and (H2) hold. Moreover, it is easy to check that (1.20) yields (1.15) and that, for every \( R \geq 0, f(\eta, \xi) \geq \xi^4/2 - M \) for every \( |\eta| \leq R \) and every \( \xi \in \mathbb{R} \) for some suitable \( M \geq 0 \) depending on \( R \).

Now, note that \( Ef^{**}(\eta, \xi) = c(\eta) \) for every \( (\eta, \xi) \in D \) so that the main hypotheses (1.13) and (1.14) of Theorem 1.2 are satisfied for instance by every smooth function \( c \) whose derivative has only isolated zeroes and which has no strict, local minima on \( \mathbb{D}^0 = \{a < 0\} \cap \{b > 0\} \).

In such case, all the hypotheses of Corollary 1.3 are satisfied and the corresponding nonconvex minimum problem \((\mathcal{P})\) has at least a solution.

**Example 1.6.** Let \( f: \mathbb{R} \times \mathbb{R} \rightarrow [0, \infty) \) be defined by

\[
f(\eta, \xi) = \begin{cases} 
\infty, & \text{for } \eta \in \mathbb{R} \text{ and } \xi \leq 0, \\
\xi - \log \xi + a(\eta)e^{-b(\eta)[\xi - c(\eta)]^2}, & \text{for } \eta \in \mathbb{R} \text{ and } \xi > 0, 
\end{cases}
\]

where the coefficients \( a, b, c \in C(\mathbb{R}) \) are positive functions. For suitable choices of \( a, b, c, f \) fails to be convex with respect to \( \xi \).

It is plain that (H1) and (H2) hold with \( C = (0, \infty) \) so that (1.17) and (1.18) obviously follow. Moreover, the growth assumption (1.19) holds too, choose \( \alpha = 1/2 \) and \( \beta = 0 \) for instance. As to (H3), note that \( Ef^{**} \) off the set \( D \) is given by

\[
Ef^{**}(\eta, \xi) = 1 - \log \xi + a(\eta) \{1 + 2b(\eta)\xi \xi - c(\eta)\} e^{-b(\eta)[\xi - c(\eta)]^2}, \quad (\eta, \xi) \notin D.
\]

Since it is easy to check that, for every \( R \geq 0 \), there is \( M = M(R) > 0 \) such that \( f(\eta, \xi) = f^{**}(\eta, \xi) \) for every \( |\eta| \leq R \) and every \( \xi \geq M \), we conclude that (H3) holds.

Now, for every \( (\eta, \xi) \in D \), there is \( \xi' > \xi \) such that \( (\eta, \xi') \in \partial D \) and \( Ef^{**}(\eta, \xi) = Ef^{**}(\eta, \xi') \).

Therefore, (1.13) holds true provided \( a, b, c \) have the appropriate behaviour, for instance, they are smooth with only isolated zeroes of the derivatives. As \( C = (0, \infty) \), the section \( \mathbb{D}^0 \) of the detachment set \( D \) is empty and the existence of solutions to the corresponding nonconvex problem \((\mathcal{P})\) follows from Corollary 1.4.

2. Some technical results

The proof of our attainment result – Theorem 1.2 – is based on the following idea: let \( z \) be a solution to the relaxed problem \((\mathcal{P}^{**})\) and let \( t \) be a differentiability point such that \( (z(t), z'(t)) \) lies in the detachment set \( D \). Then, we locally modify \( z \) around this point \( t \) thus finding a family of new solutions to \((\mathcal{P}^{**})\) which have the further property that they lie, together with their derivatives, on the “boundary” of the detachment set \( D \), i.e. where \( f \) and \( f^{**} \) coincide, almost everywhere on shrinking neighbourhoods of the point \( t \). Then, a covering argument allows to select and glue some of these new solutions so as to find a further new solution \( x \) to \((\mathcal{P}^{**})\) satisfying \( f^{**}(x, x') = f(x, x') \) almost everywhere on \([0, T]\), thus proving attainment for \((\mathcal{P})\).

The main steps towards the proof of Theorem 1.2 are gathered in this section. Indeed, the program outlined above calls first for a local description of the “boundary” of the detachment set \( D \) which is given in Proposition 2.1 below and then calls for defining new solutions to \((\mathcal{P}^{**})\) which stay
on the “boundary” of $\mathcal{D}$. These latter functions will be defined as extremal solutions to suitable, convex-valued differential inclusions related to the detachment set $\mathcal{D}$.

**Proposition 2.1.** Let $f : \mathbb{R} \times \mathbb{R} \to [0, \infty]$ be a proper and lower semicontinuous function satisfying (H1), (H2) and (H3). Then, for every $(\eta_0, \xi_0) \in \mathcal{D}$, there exists $\delta = \delta(\eta_0, \xi_0) > 0$ and two functions $a, b : [\eta_0 - \delta, \eta_0 + \delta] \to \mathbb{R}$ such that

(a) $a(\eta) < \xi_0 < b(\eta)$ for every $\eta \in [\eta_0 - \delta, \eta_0 + \delta]$;

(b) $a$ and $b$ are bounded, upper and lower semicontinuous functions respectively;

(c) $\{ (\eta, \xi) : |\eta - \eta_0| \leq \delta$ and $a(\eta) < \xi < b(\eta) \} \subset \mathcal{D}$;

(d) the equalities $f^{**}(\eta, a(\eta)) = f(\eta, a(\eta)) < \infty$ and $f^{**}(\eta, b(\eta)) = f(\eta, b(\eta)) < \infty$ hold for every $\eta \in [\eta_0 - \delta, \eta_0 + \delta]$.

With different words, (c) and (d) of Proposition 2.1 say that every connected component of every sufficiently narrow, vertical strip of $\mathcal{D}$ is the plane set contained between the graphs of two functions $a$ and $b$ satisfying (a) and (b).

**Proof of Proposition 2.1.** Let $(\eta, \xi_0)$ be a point of $\mathcal{D}$, choose $\delta = \delta(\eta_0, \xi_0) > 0$ such that $(\eta, \xi_0) \in \mathcal{D}$ when $|\eta - \eta_0| \leq \delta$ and consider the corresponding nonempty, open sections $\mathcal{D}_\eta$.

Since $\xi \in \mathcal{D}_\eta \to Ef^{**}(\eta, \xi)$ is constant on the connected components of $\mathcal{D}_\eta$ by (d) of Proposition 1.1, the growth assumption (H3) implies that $\mathcal{D}_\eta$ has bounded connected components. Hence, the functions $a$ and $b$ defined by

$$
\begin{align*}
(a) & \quad a(\eta) = \inf \{ \xi \leq \xi_0 : f^{**}(\eta, \xi') < f(\eta, \xi') \text{ for every } \xi' \in [\xi, \xi_0] \}, \\
(b) & \quad b(\eta) = \sup \{ \xi \geq \xi_0 : f^{**}(\eta, \xi') < f(\eta, \xi') \text{ for every } \xi' \in [\xi_0, \xi] \},
\end{align*}
$$

are finite and (a) and (c) hold by construction. Moreover, the open interval $(a(\eta), b(\eta))$ is the connected component of $\mathcal{D}_\eta$ containing $\xi_0$ whence the equalities in (d). Also, the closure of every connected component of $\mathcal{D}_\eta$ is contained in $C$ by (1.6) so that $f$ and $f^{**}$ are finite at the points $(\eta, a(\eta))$ and $(\eta, b(\eta))$ for every $\eta$ within $\delta$ from $\eta_0$ because of (H1).

Thus, we are left to prove (b). As regards semicontinuity, suppose for instance that $b$ fails to be lower semicontinuous at some point $\eta'$ in $[\eta_0 - \delta, \eta_0 + \delta]$ so that

$$
\liminf_{\eta \to \eta'} b(\eta) < M < b(\eta')
$$

for some real number $M > \xi_0$. It follows that $f^{**}(\eta', \xi) < f(\eta', \xi)$ for every $\xi \in [\xi_0, M]$. Hence, the compact segment $[\eta'] \times [\xi_0, M]$ is contained in the open set $\mathcal{D}$ whence $[\eta' - \sigma, \eta' + \sigma] \times [\xi_0, M]$ is in $\mathcal{D}$ too for some positive $\sigma$. This yields that $b(\eta) \geq M$ for all $\eta$ such that $|\eta - \eta'| \leq \sigma$ and $|\eta - \eta_0| \leq \delta$ and this gives a contradiction. Finally, to prove that $a$ and $b$ are bounded, note that (c) and (d) of Proposition 1.1 imply that $Ef^{**}(\eta, \xi) \geq -M$ for every $\xi \in (a(\eta), b(\eta))$ and every $\eta \in [\eta_0 - \delta, \eta_0 + \delta]$ for some $M \geq 0$ and that the growth assumption (H3) yields that

$$
\sup \{ Ef^{**}(\eta, \xi) : |\eta - \eta_0| \leq \delta \} < -M, \quad |\xi| \geq R,
$$

for some large enough $R$. Thus, $-R < a(\eta) < b(\eta) < R$ for every $|\eta - \eta_0| \leq \delta$ and this completes the proof.

The following lemma is proved in [2]. It is a technical result whose statement is long though its proof is elementary.

**Lemma 2.2.** Let $z \in AC([0, T])$ be differentiable at some point $s \in (0, T)$ and let $\alpha, \beta \in \mathbb{R}$ be such that

$$
\alpha < z'(s) < \beta.
$$
Then, for every $\delta > 0$, there exist $\varepsilon_0 = \varepsilon_0(s, \delta) > 0$, two families of compact subintervals \( \{ H_{s, \varepsilon}^+ : 0 < \varepsilon \leq \varepsilon_0 \} \) of \((0, T)\) and two families of functions \( \{ z_{s, \varepsilon}^\pm : 0 < \varepsilon \leq \varepsilon_0 \} \) in \( AC([0, T]) \) such that, setting

\[
J_{s, \varepsilon}^+ = \left( s - \frac{\varepsilon}{\beta - z'(s)} , s + \frac{\varepsilon}{\beta - z'(s)} \right), \quad J_{s, \varepsilon}^- = \left( s - \frac{\varepsilon}{\beta - z'(s)} , s + \frac{\varepsilon}{\beta - z'(s)} \right),
\]

for every $\varepsilon > 0$, the following properties hold:

\[
(2.1) \quad \varepsilon < \varepsilon_0(s, \delta) > 0,
\]

\[
(2.2) \quad J_{s, \varepsilon}^+ \subset H_{s, \varepsilon}^+ \subset J_{s, 2\varepsilon}^+ \subset (0, T);
\]

\[
(2.3) \quad z_{s, \varepsilon}^+ = z \text{ on } [0, T] \setminus \text{int } (H_{s, \varepsilon}^+);
\]

\[
(2.4+) \quad z(t) < z_{s, \varepsilon}^+(t) \leq z(t) + \delta \text{ for every } t \in \text{int } (H_{s, \varepsilon}^+);
\]

\[
(2.4-) \quad z(t) - \delta \leq z_{s, \varepsilon}^-(t) < z(t) \text{ for every } t \in \text{int } (H_{s, \varepsilon}^-);
\]

\[
(2.5+) \quad \varepsilon \geq z_{s, \varepsilon}^+(t) - [z(s) + z'(s)(t - s)] \geq \varepsilon/2 \text{ for every } t \in J_{s, \varepsilon}^+/2;
\]

\[
(2.5-) \quad -\varepsilon/2 \geq z_{s, \varepsilon}^-(t) - [z(s) + z'(s)(t - s)] \geq -\varepsilon \text{ for every } t \in J_{s, \varepsilon}^-/2;
\]

\[
(2.6) \quad \left( z_{s, \varepsilon}^\prime \right)^+(t) \in \{ \alpha, \beta \} \text{ for a.e. } t \in H_{s, \varepsilon}^+;
\]

hold for every $0 < \varepsilon \leq \varepsilon_0$.

Note in particular that, setting

\[
E = \{ s \in (0, T) : z \text{ is differentiable at } s \text{ with } \alpha < z'(s) < \beta \},
\]

the family of all compact sets \( \{ H_{s, \varepsilon}^+ : 0 < \varepsilon \leq \varepsilon_0(s, \delta), s \in E \} \) is a Vitali covering of the measurable set $E$ itself.

Then, we construct the comparison conditions that will be used in the proof of Theorem 1.2. As mentioned above, they will be defined as extremal solutions to suitable, convex-valued differential inclusions.

**Proposition 2.3.** Let $a, b : [\eta_0 - \delta, \eta_0 + \delta] \rightarrow \mathbb{R}$ be two bounded, upper and lower semicontinuous functions such that

(a) $a(\eta) < \Xi_0 < b(\eta)$ for every $\eta \in [\eta_0 - \delta, \eta_0 + \delta]$;

for some $\Xi_0 \in \mathbb{R}$ and assume that there exist $y \in AC([0, T])$ and $t_0 \in (0, T)$ such that

(b) $y$ is differentiable at $t_0$ with $y(t_0) = \eta_0$ and $y'(t_0) = \Xi_0$.

Then, there exist $\varepsilon_0 = \varepsilon_0(t_0, \delta) > 0$, two families of compact subintervals \( \{ K_{t_0, \varepsilon}^\pm : 0 < \varepsilon \leq \varepsilon_0 \} \) of \((0, T)\) such that

\[
(2.7) \quad \text{each set } K_{t_0, \varepsilon}^\pm \text{ is a neighbourhood of } t_0 \text{ and } \{ K_{t_0, \varepsilon}^\pm : 0 < \varepsilon \leq \varepsilon_0 \} \text{ shrinks at } t_0;
\]

and two families of functions \( \{ y_{t_0, \varepsilon}^\pm : 0 < \varepsilon \leq \varepsilon_0 \} \) in \( AC([0, T]) \) such that the following properties

\[
(2.8) \quad y_{t_0, \varepsilon}^\pm = y \text{ on } [0, T] \setminus \text{int } (K_{t_0, \varepsilon}^\pm);
\]

\[
(2.9+) \quad y(t) < y_{t_0, \varepsilon}^+(t) \leq y(t) + \varepsilon \text{ for every } t \in \text{int } (K_{t_0, \varepsilon}^+);
\]

\[
(2.9-) \quad y(t) - \varepsilon \leq y_{t_0, \varepsilon}^-(t) < y(t) \text{ for every } t \in \text{int } (K_{t_0, \varepsilon}^-);
\]

\[
(2.11) \quad |y_{t_0, \varepsilon}^\pm(t) - \eta_0| \leq \delta \text{ for every } t \in K_{t_0, \varepsilon}^\pm;
\]

\[
(2.12) \quad \left(y_{t_0, \varepsilon}^\pm\right)^\prime(t) \in \{ a\left(y_{t_0, \varepsilon}^\pm(t)\right), b\left(y_{t_0, \varepsilon}^\pm(t)\right)\} \text{ for a.e. } t \in K_{t_0, \varepsilon}^\pm;
\]
hold for every $0 < \varepsilon \leq \varepsilon_0$.

**Proof.** We are going to treat the + case, the other one being entirely equivalent. Thereby, to simplify the notations, we shall drop the superscript + from now on. Our strategy is the following: relying on Lemma 2.2, for every small enough $\varepsilon$, we are going to define the compact set $K_{t_0,\varepsilon}$ and a sequence of functions $y_{k,t_0,\varepsilon}$ in $AC([0,T])$ satisfying (2.8), (2.9+) and (2.11) which are "approximated" solutions to the differential inclusion (2.12). The remarkable point is that, though the derivatives of the "approximated" solutions $y_{k,t_0,\varepsilon}$ oscillate faster and faster as $k \to \infty$ in order to solve (2.12), the functions $y_{k,t_0,\varepsilon}$ can be defined in such a way that they do converge strongly in $AC([0,T])$. The limit function will be $y_{t_0,\varepsilon}$.

To this purpose, set $I = [\eta_0 - \delta, \eta_0 + \delta]$ to simplify the notations and define

$$\begin{align*}
M &= \max \{a(\eta) : \eta \in I\} \\
\sigma &= \min \{m - \xi_0, \xi_0 - M\}
\end{align*}$$

(2.13)

so that $\sigma > 0$ because of (a). Then, choose $\varepsilon_0 = \varepsilon_0'(t_0, \delta, \sigma) > 0$ small enough so that the following

$$\begin{align*}
\varepsilon_0' < \min \{\delta/4, \sigma/16, \sigma T_0, \sigma (T - t_0)/4\} \\
|t - t_0| \leq 4\varepsilon_0'/\sigma \quad \Rightarrow \quad |y(t) - \eta_0| < \min \{\delta/4, \sigma/32\}
\end{align*}$$

(2.14)

hold and define also

$$\begin{align*}
a_k(\eta) &= \max \{a(\eta') - k |\eta' - \eta| : \eta' \in I\} + \sigma/2k \\
b_k(\eta) &= \min \{b(\eta') + k |\eta' - \eta| : \eta' \in I\} - \sigma/2k
\end{align*}$$

(2.15)

These functions $a_k$ and $b_k$ are (up to the constants $\sigma/2k$) the Moreau-Yosida approximations of $a$ and $b$ respectively. They enjoy the following properties (see [10] for instance):

(2.16) $a_k$ and $b_k$ are Lipschitz continuous on $I$ with Lipschitz constant $k$;

(2.17) $M + \sigma/2 \geq a_1(\eta)$ and $a_k(\eta) - a_{k+1}(\eta) \geq \Delta_k = \sigma/2k(k+1)$ for every $\eta \in I$ and $k \geq 1$;

(2.18) $m - \sigma/2 \leq b_1(\eta)$ and $b_k(\eta) - b_{k+1}(\eta) \geq \Delta_k = \sigma/2k(k+1)$ for every $\eta \in I$ and $k \geq 1$;

(2.19) $a_k \to a$ and $b_k \to b$ pointwise on $I$.

Now, the preparatory work is over and the proof will be completed by proving the following three claims.

**Claim 1.** There exist $\varepsilon_0 = \varepsilon_0(t_0, \delta, \sigma) \in (0, \varepsilon_0']$ and a family of compact subintervals $\{K_{t_0,\varepsilon} : 0 < \varepsilon \leq \varepsilon_0\}$ of $(0, T)$ satisfying (2.7) with the further property that, for every sequence of positive numbers $\{\omega_k\}_{k \geq 1}$, the following holds for every $\varepsilon \in (0, \varepsilon_0]$: there exists a sequence of functions $\{y_{k,t_0,\varepsilon}\}_{k \geq 0}$ in $AC([0,T])$ such that $y_{0,t_0,\varepsilon} = y$ and

$$\begin{align*}
y_{k,t_0,\varepsilon} &= y \text{ on } [0,T] \setminus \text{int } (K_{t_0,\varepsilon}); \\
0 < y_{k,t_0,\varepsilon}(t) - y_{k-1,t_0,\varepsilon}(t) < \min \{\varepsilon, \omega_k, \frac{\Delta_k}{4(k+1)}\} & \text{ for every } t \in (K_{t_0,\varepsilon}); \\
|y_{k,t_0,\varepsilon}(t) - \eta_0| < \sum_{1 \leq h \leq k} \delta/2^h & \text{ for every } t \in K_{t_0,\varepsilon}; \\
y'_{k,t_0,\varepsilon}(t) &\in (a_{k+1}(y_{k,t_0,\varepsilon}(t)), a_k(y_{k,t_0,\varepsilon}(t))) \cup (b_k(y_{k,t_0,\varepsilon}(t)), b_{k+1}(y_{k,t_0,\varepsilon}(t))) & \text{ for a.e. } t \in K_{t_0,\varepsilon};
\end{align*}$$

(2.20) (2.21) (2.22) (2.23)

for every $k \geq 1$. 


Claim 2. There exists a sequence of positive numbers \( \{ \omega_k \}_k \) such that, for every \( \varepsilon \in (0, \varepsilon_0] \), the sequence of functions \( \{ y_{k,t_0,\varepsilon} \}_k \) converges strongly in \( AC([0, T]) \) to a function \( y_{t_0,\varepsilon} \in AC([0, T]) \).

Claim 3. For the same sequence of numbers \( \{ \omega_k \}_k \), we also have that

\[
(2.24) \quad \begin{cases} 
 a_{k+h}(y_{k,t_0,\varepsilon}(t)) \to a(y_{t_0,\varepsilon}(t)) \\
 b_{k+h}(y_{k,t_0,\varepsilon}(t)) \to b(y_{t_0,\varepsilon}(t))
\end{cases}
\text{as } k \to \infty
\]

for every \( h \geq 0 \) and \( t \in [0, T] \).

Once the previous claims have been proved, we conclude immediately that \( y_{t_0,\varepsilon} \) satisfies (2.8), (2.9+) and (2.11) for every \( \varepsilon \in (0, \varepsilon_0] \) because of the corresponding properties of the functions \( y_{k,t_0,\varepsilon} \). Moreover, Claim 2 implies that some subsequence of \( \{ y'_{k,t_0,\varepsilon} \}_k \) converges to \( y'_{t_0,\varepsilon} \) almost everywhere on \([0, T]\) so that (2.12) follows from (2.23) and Claim 3.

**Proof of Claim 1.** We apply Lemma 2.2 in the + case with \( s = t_0, z = y \) and \( \alpha = [a_1(\eta_0) + a_2(\eta_0)]/2 \) and \( \beta = [b_1(\eta_0) + b_2(\eta_0)]/2 \) thus finding \( \varepsilon_0 = \varepsilon_0(t_0, \delta, \sigma) \in (0, \varepsilon_0] \), a family of compact subintervals \( \{ H_{t_0,\varepsilon} \} : 0 < \varepsilon \leq \varepsilon_0 \} \) of \([0, T)\) which are all neighbourhoods of \( t_0 \) and a family of functions \( \{ z_{t_0,\varepsilon} : 0 < \varepsilon \leq \varepsilon_0 \} \) in \( AC([0, T]) \) such that (2.2), (2.3), (2.4+) and (2.6) hold. Relying on these properties, it is easy to check that \( (z_{t_0,\varepsilon} - y) \to 0_+ \) uniformly on \([0, T)\) as \( \varepsilon \to 0 \) so that, for every \( 0 < \varepsilon \leq \varepsilon_0 \), we can choose \( 0 < \varepsilon' \leq \varepsilon \) such that

\[
0 < \varepsilon' < \varepsilon \quad \text{and} \quad \langle \varepsilon, \varepsilon' \rangle < \varepsilon_0.
\]

Since \( \varepsilon' = \varepsilon_0 - \varepsilon_0 \alpha/2 \), this latter inclusion, together with the equality \( y_{t_0,\varepsilon} = y \) for every \( \varepsilon \in (0, \varepsilon_0] \) by definition and setting \( K_{t_0,\varepsilon} = H_{t_0,\varepsilon} \) and \( y_{1, t_0,\varepsilon} = z_{t_0,\varepsilon} \) for every \( \varepsilon \in (0, \varepsilon_0] \), we conclude that \( K_{t_0,\varepsilon} \) satisfies (2.7), that \( y_{1, t_0,\varepsilon} \) satisfies (2.20) and (2.21) with \( k = 1 \) and moreover that the derivative of \( y_{1, t_0,\varepsilon} \) is such that

\[
y'_{1, t_0,\varepsilon}(t) \in \left\{ \frac{a_1(\eta_0) + a_2(\eta_0)}{2}, \frac{b_1(\eta_0) + b_2(\eta_0)}{2} \right\}
\]

for a.e. \( t \in K_{t_0,\varepsilon} \).

As regards (2.22), note that each compact interval \( K_{t_0,\varepsilon} \) is contained in \( H_{t_0,\varepsilon_0} \) and that this latter interval is contained in \([t_0 - 4\varepsilon_0/\sigma, t_0 + 4\varepsilon_0/\sigma] \) by (2.2). This latter inclusion, together with the equality \( y_{t_0,\varepsilon} = y \) and the choice of \( \varepsilon_0 \) made in (2.14), implies that

\[
(2.25) \quad |y_{t_0,\varepsilon}(t) - \eta_0| \leq \min \left\{ \frac{\varepsilon}{2}, \frac{\Delta_1}{4(1 + 1)} \right\}
\]

since \( \sigma/32 = \Delta_1/8 \). Hence,

\[
(2.26) \quad |y_{1, t_0,\varepsilon}(t) - \eta_0| \leq |y_{1, t_0,\varepsilon}(t) - y_{t_0,\varepsilon}(t)| + |y_{t_0,\varepsilon}(t) - \eta_0| < \varepsilon/2 + \delta/4 \leq \delta/4 + \delta/4 = \delta/2
\]

holds for every \( t \in K_{t_0,\varepsilon} \) because of (2.21) with \( k = 1 \) and (2.25). At last, to complete the first step, we have to check that \( y_{1, t_0,\varepsilon} \) is an “approximated” solution to the differential inclusion (2.12) on the set \( K_{t_0,\varepsilon} \), i.e. that (2.23) holds for \( k = 1 \). Indeed, let \( y_{1, t_0,\varepsilon} \) be differentiable at some point \( t \in \text{int}(K_{t_0,\varepsilon}) \) with derivative \( y'_{1, t_0,\varepsilon}(t) = [a_1(\eta_0) + a_2(\eta_0)]/2 \). By elementary computations based on (2.16), (2.17), (2.21) for \( k = 1 \) and (2.25), we get

\[
a_2(y_{1, t_0,\varepsilon}(t)) \leq a_2(\eta_0) + 2 \left( |y_{1, t_0,\varepsilon}(t) - y_{t_0,\varepsilon}(t)| + |y_{t_0,\varepsilon}(t) - \eta_0| \right) < a_2(\eta_0) + \frac{\Delta_1}{2} = \frac{a_1(\eta_0) + a_2(\eta_0)}{2}
\]

and similarly

\[
a_1(y_{1, t_0,\varepsilon}(t)) > \frac{a_1(\eta_0) + a_2(\eta_0)}{2}.
\]
Thus, $a_2(y_{1,t_0,\varepsilon}(t)) < y'_{1,t_0,\varepsilon}(t) < a_1(y_{1,t_0,\varepsilon}(t))$ and the very same kind of computations in the case that $y'_{1,t_0,\varepsilon}(t) = [b_1(\eta_0) + b_2(\eta_0)]/2$ yield that $b_1(y_{1,t_0,\varepsilon}(t)) < y'_{1,t_0,\varepsilon}(t) < b_2(y_{1,t_0,\varepsilon}(t))$ which is (2.23) for $k = 1$.

Next, we go on defining the second “approximated” solution $y_{2,t_0,\varepsilon}$ on the same set $K_{t_0,\varepsilon}$. We shall do this for a fixed $\varepsilon \in (0, \varepsilon_0]$.

To this purpose, choose any point $t \in \text{int} (K_{t_0,\varepsilon})$ where $y_{1,t_0,\varepsilon}$ is differentiable and (2.23) holds for $k = 1$ and set $\eta_t = y_{1,t_0,\varepsilon}(t)$. For every such point $t$, we apply Lemma 2.2 in the case with $s = t$, $z = y_{1,t_0,\varepsilon}$ and $\alpha = [a_2(\eta_t) + a_3(\eta_t)]/2$ and $\beta = [b_2(\eta_t) + b_3(\eta_t)]/2$, thus finding a positive number $\theta_0 = \theta_0(t, \varepsilon)$, a family of nondegenerate, compact intervals $\{L_{t,\theta} : 0 < \theta \leq \theta_0\}$ contained in $K_{t_0,\varepsilon}$ and a family of functions $\{z_{t,\theta} : 0 < \theta \leq \theta_0\}$ in $AC([0, T])$ such that all sets $L_{t,\theta}$ are neighbourhoods of $t$ and the following properties

\[ z_{t,\theta} = y_{1,t_0,\varepsilon} \]  
\[ 0 < z_{t,\theta}(s) - y_{1,t_0,\varepsilon}(s) \leq \min \left\{ \varepsilon, \frac{\Delta_2}{2(2 + 1)} \right\} \]  
\[ z'_{t,\theta}(s) \in \left\{ \frac{a_2(\eta_t) + a_3(\eta_t)}{2}, \frac{b_2(\eta_t) + b_3(\eta_t)}{2} \right\} \]  

hold for every $\theta \in (0, \theta_0]$. Moreover, we can assume that $\theta_0$ is small enough to have

\[ |y_{1,t_0,\varepsilon}(s) - \eta_t| \leq \frac{\Delta_2}{4(2 + 1)} \]  

for every $s \in L_{t,\theta}$ and $\theta \in (0, \theta_0]$.

and we note also that, from (2.28), (2.14) and (2.26), it follows that

\[ |z_{t,\theta}(s) - \eta_t| \leq |z_{t,\theta}(s) - y_{1,t_0,\varepsilon}(s)| + |y_{1,t_0,\varepsilon}(s) - \eta_t| \leq \delta/2 + \delta/4 \]  

for every $s \in L_{t,\theta}$ and $\theta \in (0, \theta_0]$. Next, we prove that

\[ z'_{t,\theta}(s) \in \left( a_2(z_{t,\theta}(s)) \cup a_3(z_{t,\theta}(s)) \right) \cup \left( a_2(z_{t,\theta}(s)) \cup a_3(z_{t,\theta}(s)) \right) \]  

for a.e. $s \in L_{t,\theta}$. Indeed, (2.28), (2.17) and (2.30) yield

\[ a_3(z_{t,\theta}(s)) \leq a_3(\eta_t) + 3 \left( |z_{t,\theta}(s) - y_{1,t_0,\varepsilon}(s)| + |y_{1,t_0,\varepsilon}(s) - \eta_t| \right) < a_3(\eta_t) + \frac{\Delta_2}{2} \leq \frac{a_2(\eta_t) + a_3(\eta_t)}{2} \]  

and similarly

\[ \frac{a_2(\eta_t) + a_3(\eta_t)}{2} < a_3(z_{t,\theta}(s)) \]

for a.e. $s \in L_{t,\theta}$. Thus, (2.32) holds if the derivative of $z_{t,\theta}$ exists and is equal to $[a_2(\eta_t) + a_3(\eta_t)]/2$ and a specular argument yields (2.32) if $z'_{t,\theta}(s) = [b_2(\eta_t) + b_3(\eta_t)]/2$.

So far, we have defined a family of functions $z_{t,\theta}$ satisfying (2.20), (2.21), (2.22) and (2.23) for $k = 2$ around every “good” point $t$ in the interior of $K_{t_0,\varepsilon}$ and, to complete the definition of $y_{2,t_0,\varepsilon}$, we just have to glue these functions $z_{t,\theta}$ by a covering argument. Indeed, by the remark following Lemma 2.2, the family of nondegenerate, compact intervals $\{L_{t,\theta} : 0 < \theta \leq \theta_0(t, \varepsilon)\}$ constitutes a Vitali covering of the measurable set

\[ E_1 = \{ t \in \text{int} (K_{t_0,\varepsilon}) : y'_{1,t_0,\varepsilon}(t) \text{ exists and (2.23) holds with } k = 1 \} \]

which is a full measure subset of $\text{int} (K_{t_0,\varepsilon})$. Hence, Vitali’s covering theorem yields (at most) countably many points $t_j \in E_1$ and positive numbers $\theta_j \in (0, \theta_0(t_j, \varepsilon)]$ such that the corresponding compact intervals $L_j = L_{t_j,\theta_j}$ are pairwise disjoint sets that cover $E_1$, and hence $K_{t_0,\varepsilon}$ as well, up to a null set.
Now, we define the second “approximated” solution $y_{2,t_0,\varepsilon}$ by setting
\begin{equation}
(2.33) \quad y_{2,t_0,\varepsilon}(t) = y_{1,t_0,\varepsilon}(t) + \sum_{j \geq 1} [z_j(t) - y_{1,t_0,\varepsilon}(t)], \quad t \in [0, T],
\end{equation}
where $z_j = z_{t_j,\beta_j}$. As the supports $L_j$ of the functions $z_j - y_{1,t_0,\varepsilon}$ are disjoint, the series in (2.33) is actually a finite sum for every $t$ and moreover, the functions $z_j - y_{1,t_0,\varepsilon}$ have (essentially) uniformly bounded derivatives on $[0, T]$. Thus, $y_{2,t_0,\varepsilon}$ is Lipschitz continuous on $[0, T]$ by Ascoli-Arzelà’s theorem and the fulfillment of (2.20), (2.21), (2.22) and (2.23) for $k = 2$ follows straightforwardly from the corresponding properties (2.27), (2.28), (2.31) and (2.32) of the functions $z_j = z_{t_j,\beta_j}$.

Finally, all the remaining functions $y_{k,t_0,\varepsilon}$ are defined recursively in the very same way we have got $y_{2,t_0,\varepsilon}$ from $y_{1,t_0,\varepsilon}$ and this completes the proof of Claim 1.

**Proof of Claim 2.** We are going to choose the positive numbers $\omega_k$ so as to have strong convergence in $L^1([0, T])$ of the derivatives $y_{k,t_0,\varepsilon}$.

To this purpose, let $\varphi \in \mathcal{D}(\mathbb{R})$ be the standard mollifying kernel and set, as usual, $\varphi_r(t) = r^{-1} \varphi(t/r)$ for every $t \in \mathbb{R}$ and $r > 0$. Choose also a sequence of positive numbers $0 < r_k < 2^{-k}$ in such a way that, extending each function $y_{k,t_0,\varepsilon}$ to the whole real line as a constant function off the interval $[0, T]$, the following inequality holds
\begin{equation}
(2.34) \quad \int_{\mathbb{R}} |\varphi_{r_k} * y_{k,t_0,\varepsilon}(t) - y'_{k,t_0,\varepsilon}(t)| \, dt \leq \frac{1}{k}, \quad k \geq 1.
\end{equation}

Then, let $\{\omega_k\}_k$ be the sequence defined by setting $\omega_1 = 1$ and recursively
\begin{equation}
(2.35) \quad \omega_{k+1} = r_k \omega_k, \quad k \geq 1.
\end{equation}

The reader might think that this way of choosing the numbers $\omega_k$ is inconsistent as it requires that the functions $y_{k,t_0,\varepsilon}$ be already defined. This is not the case. Indeed, we set $\omega_1$ to be 1 and then we define $y_{1,t_0,\varepsilon}$ so that (2.21) with $k = 1$ holds. Then, we compute $r_k$ according to (2.34) with $k = 1$ – which requires only that $y_{1,t_0,\varepsilon}$ be defined – and then we define the number $\omega_2 = r_1 \omega_1$. Only then, we choose the function $y_{2,t_0,\varepsilon}$ so that (2.21) with $k = 2$ holds and we restart the procedure.

Now, we claim that this choice of the numbers $\omega_k$ yields the conclusion of Claim 2 and we break the remaining part of the proof in the following three claims.

**Claim 2.1.** For every $\varepsilon \in (0, \varepsilon_0]$, the sequence $\{y_{k,t_0,\varepsilon}\}_k$ converges uniformly on $[0, T]$ to some function $y_{t_0,\varepsilon}$.

Indeed,
\begin{equation}
(2.36) \quad 0 \leq y_{k+1,t_0,\varepsilon}(t) - y_{k,t_0,\varepsilon}(t) \leq \omega_{k+1}, \quad t \in [0, T]
\end{equation}
by (2.20) and (2.21). Since $0 < \omega_{k+1}/\omega_k = r_k \to 0$ as $k \to \infty$, the sequence $\{y_{k,t_0,\varepsilon}\}_k$ is uniformly Cauchy on $[0, T]$ and the conclusion follows.

**Claim 2.2.** $y_{t_0,\varepsilon} \in AC([0, T])$.

All the functions $y_{k,t_0,\varepsilon}$ have (essentially) uniformly bounded derivatives on the interval $K_{t_0,\varepsilon}$ since $a$ and $b$ are bounded and
\[ a(y_{k,t_0,\varepsilon}(t)) < a_{k+1}(y_{k,t_0,\varepsilon}(t)) < b_{k+1}(y_{k,t_0,\varepsilon}(t)) < b(y_{k,t_0,\varepsilon}(t)), \quad \text{for a.e. } t \in K_{t_0,\varepsilon}. \]

Thus, $y_{t_0,\varepsilon}$ is Lipschitz continuous on $K_{t_0,\varepsilon}$. As it coincides with the absolutely continuous function $y$ on $[0, T] \setminus \text{int}(K_{t_0,\varepsilon})$ because of (2.20), the claim is proved.

**Claim 2.3.** The sequence $\{y_{k,t_0,\varepsilon}\}_k$ converges strongly in $AC([0, T])$.

Indeed,
\[ \|y'_{k,t_0,\varepsilon} - y'_{t_0,\varepsilon}\|_1 \leq \frac{1}{k}. \]
\[ \eta(t,0,\varepsilon) - \varphi_{rk} \ast y_{k,t_0,\varepsilon} \leq \|\varphi_{rk} \ast y_{k,t_0,\varepsilon} - \varphi_{rk} \ast y_{t_0,\varepsilon}\|_1 + \|\varphi_{rk} \ast y_{t_0,\varepsilon} \ast \varphi_{rk} \ast y_{t_0,\varepsilon}\|_1. \]

The first and the third summand at the right hand side go to zero as \( k \to \infty \) because of (2.34) and the properties of convolutions with mollifying kernels respectively. We are thus left to prove that

\[ R_k = \|\varphi_{rk} \ast (y_{k,t_0,\varepsilon} - y_{t_0,\varepsilon})\|_1 \to 0 \quad \text{as} \quad k \to \infty. \]

To this aim, note that \( R_k = C r_k^{-1} \|y_{k,t_0,\varepsilon} - y_{t_0,\varepsilon}\|_\infty \) where \( C = T \|\varphi'\|_1 \) so that (2.35) and (2.36) yield that

\[ \|y_{k,t_0,\varepsilon} - y_{t_0,\varepsilon}\|_\infty \leq \sum_{j \geq 1} \|y_{k+j,t_0,\varepsilon} - y_{k+j-1,t_0,\varepsilon}\|_\infty \leq \sum_{j \geq 1} \omega_{k+j} = \omega_{k+1} (1 + \sum_{j \geq 1} r_{k+j}) \leq 2\omega_{k+1} = 2r_k \omega_k \]

because \( 0 < r_k < 2^{-k} \) by assumption. Thus, \( R_k \leq 2C \omega_k \to 0 \) as \( k \to \infty \) and the conclusion follows.

**Proof of Claim 3.** Let \( t \) be in the interior of \( K_{t_0,\varepsilon} \) otherwise the conclusion follows immediately from (2.20) and (2.19). For such \( t \), we have

\[ |a_{k+h}(y_{k,t_0,\varepsilon}(t)) - a(y_{t_0,\varepsilon}(t))| \leq |a_{k+h}(y_{k,t_0,\varepsilon}(t)) - a_{k+h}(y_{t_0,\varepsilon}(t))| + |a_{k+h}(y_{t_0,\varepsilon}(t)) - a(y_{t_0,\varepsilon}(t))| \]

and the second summand at the right hand side goes to zero as \( k \to \infty \) because of (2.19) again, no matter what \( h \) is. As to the first one, (2.16) and the very same argument of Claim 2.3 show that it is bounded by \( 2(k+h)\omega_{k+1} \) and this goes to zero as \( k \to \infty \) because (2.35) and the basic assumption \( 0 < r_k < 2^{-k} \) yield that \( \omega_{k+1} \leq 2^{-k} \). This proves the \( a \)-case and nothing changes in the \( b \)-case.

3. PROOF OF THE MAIN RESULT

In this final section, we put together and exploit the tools developed in the previous section and prove our attainment result, Theorem 1.2.

**Proof of Theorem 1.2.** Let \( R \) be a bounded, open rectangle whose closure is contained in \( D \). By (c) and (d) of Proposition 1.1, the function

\[ q(\eta) = Ef^*(\eta,\xi), \quad (\eta,\xi) \in \overline{R}, \]

is well defined and we claim that it has at most finitely many strict, local extrema in \( \overline{R} \). Indeed, should this be false, there would be a converging sequence \( \{m_k\}_k \) of strict, local extrema for \( q \), say \( m_k \to m_0 \), and we could assume also the sequence \( \{m_k\}_k \) is strictly monotone. Thus, \( q \) would fail to be monotone on both sides of \( m_0 \) and this gives a contradiction to (1.13). As \( D \) is a countable union of such rectangles \( R \), we conclude that there exists a (at most) countable family of subsets of \( D \), say \( \{m_i\}_i \times J_i \), with the property that, for every index \( i \), \( J_i \) is a connected component of \( D_{m_i} \) and, for every \( \xi \in J_i \), \( m_i \) is a strict, local extremum point for the mapping \( \eta \in D^k \to Ef^*(\eta,\xi) \). Conversely, if \( (\eta,\xi) \in D \) is such that \( \eta \) is a strict, local extremum point for \( \eta \in D^k \to Ef^*(\eta,\xi) \), then \( \eta = m_i \) for some index \( i \) and the corresponding open interval \( J_i \) is the connected component of \( D_{m_i} \) containing \( \xi \). We recall also that, according to (e) of Proposition 1.1 and (1.14), a point \( m_i \) may be a strict, local minimum point for \( \eta \in D^k \to Ef^*(\eta,\xi) \) for some \( \xi \in J_i \) only if \( 0 \notin J_i \).

Now, let \( y \in AC([0,T]) \) be a solution to \((P^*)\) and assume that \( I^*(y) < \infty \) otherwise there is nothing else to prove. We are going to prove that \( y \) can be modified so as to find a new solution \( x \) to \((P^*)\) such that

\[ f^*(x(t),x'(t)) = f(x(t),x'(t)) \quad \text{for a.e.} \ t \in [0,T], \]

thus showing that \( x \) is a solution to \((P)\) as well. The proof goes through the following three steps.
Step 1. Let $M$ be the subset of $\mathcal{D}$ defined by $M = \cup_{i} \left( \{ m_i \} \times J_i \right)$ and note that $\mathcal{D} \setminus M$ is open. First, we prove that, whenever the measurable set
\begin{equation}
E = \left\{ t \in (0, T) : \text{y is differentiable at } t \text{ and } \left( y(t), y'(t) \right) \in \mathcal{D} \setminus M \right\}
\end{equation}
has positive measure, we can use Lemma 2.2 and Proposition 2.3 to associate with almost every point $s \in E$ a family of new solutions $\{ y_{s, \varepsilon} : 0 < \varepsilon \leq \varepsilon_0(s) \}$ to $(P^{**})$ such that the sets $K_{s, \varepsilon}$ defined as the closure of $\{ y_{s, \varepsilon} \neq y \}$ are nondegenerate, compact intervals that shrink at $s$ and the following properties
\begin{align}
(3.3a) & \quad \sup \{ |y_{s, \varepsilon}(t) - y(t)| : 0 \leq t \leq T \} \leq \varepsilon; \\
(3.3b) & \quad f^{**}(y_{s, \varepsilon}(t), y_{s, \varepsilon}(t)) = f(y_{s, \varepsilon}(t), y_{s, \varepsilon}(t)) \text{ for a.e. } t \in K_{s, \varepsilon};
\end{align}
hold for every $0 < \varepsilon \leq \varepsilon_0(s)$.

Step 2. Then, we use the modified solutions of the previous step and a covering argument to define a new solutions $x$ to $(P^{**})$ such that, setting
\begin{equation}
A = \left\{ t \in (0, T) : x \text{ is differentiable at } t \text{ and } \left( x(t), x'(t) \right) \in M \right\},
\end{equation}
we have that
\begin{equation}
f^{**}(x(t), x'(t)) = f(x(t), x'(t)) \quad \text{for a.e. } t \in [0, T] \setminus A.
\end{equation}

Step 3. Finally, we show that $A$ is negligible. Thus, (3.5) reduces to (3.1) and this shows that $x$ is a solution to $(P)$.

**Proof of Step 1.** Assume that the set $E$ defined by (3.2) has positive measure, fix a point $t_0 \in E$ and set $\eta_0 = y(t_0)$ and $\xi_0 = y'(t_0)$. As $\eta_0, \xi_0 \in \mathcal{D} \setminus M$ by assumption, the basic hypotheses (1.13) and (1.14) and the very definition of the set $M$ itself imply that the restriction $\eta \in D^{\eta_0} \to Ef^{**}(\eta, \xi_0)$ is monotone on the interval $[\eta_0 - \delta, \eta_0 + \delta]$ for some $\delta > 0$. Moreover, by possibly choosing a smaller value of $\delta$, we can describe the upper and the lower parts of the boundary of the connected component of the vertical strip $\mathcal{D} \cap \left( [\eta_0 - \delta, \eta_0 + \delta] \times \mathbb{R} \right)$ of $\mathcal{D}$ containing $(\eta_0, \xi_0)$ as in Proposition 2.1, i.e. as the graphs of the functions $a, b: [\eta_0 - \delta, \eta_0 + \delta] \to \mathbb{R}$ satisfying (a), (b), (c) and (d) of Proposition 2.1. Also, setting $\mathcal{D}' = \{ (\eta, \xi) : |\eta - \eta_0| \leq \delta \text{ and } a(\eta) \leq \xi \leq b(\eta) \}$ and recalling (1.5), (1.6) and Proposition 1.1, we can write the convex envelope $f^{**}$ of $f$ on the set $\mathcal{D}'$ as
\begin{equation}
f^{**}(\eta, \xi) = d(\eta)\xi + q(\eta), \quad (\eta, \xi) \in \mathcal{D}',
\end{equation}
where the continuous functions $d, q: [\eta_0 - \delta, \eta_0 + \delta] \to \mathbb{R}$ are defined by $q(\eta) = Ef^{**}(\eta, \xi_0)$ and $d(\eta) = d(\eta, \xi)$ for every $(\eta, \xi) \in \mathcal{D}'$. Note also that $q$ is monotone on the interval $[\eta_0 - \delta, \eta_0 + \delta]$ because of the corresponding property of the restriction $\eta \in D^{\eta_0} \to Ef^{**}(\eta, \xi_0)$, that
\begin{equation}
f^{**}(\eta, \xi) \geq d(\eta)\xi + q(\eta), \quad \text{for every } \eta \in [\eta_0 - \delta, \eta_0 + \delta] \text{ and } \xi \in \mathbb{R}
\end{equation}
holds because of (1.1) and (1.7) and that the equalities
\begin{equation}
\left\{ \begin{array}{ll}
f^{**}(\eta, a(\eta)) = f(\eta, a(\eta)) \\
f^{**}(\eta, b(\eta)) = f(\eta, b(\eta))
\end{array} \right. \quad \eta \in [\eta_0 - \delta, \eta_0 + \delta],
\end{equation}
follow from (d) of Proposition 2.1. Then, we apply Proposition 2.3 and we let $\{ K_{t_0, \varepsilon}^\pm : 0 < \varepsilon \leq \varepsilon_0(t_0) \}$ and $\{ y_{t_0, \varepsilon}^\pm : 0 < \varepsilon \leq \varepsilon_0(t_0) \}$ be the corresponding intervals and functions and we assume that $\varepsilon_0(t_0)$ is small enough so as to have
\begin{equation}
|y(t) - \eta_0| \leq \delta, \quad \text{for every } t \in K_{t_0, \varepsilon}^\pm \text{ and } 0 < \varepsilon \leq \varepsilon_0(t_0).
\end{equation}
Now, we wish to compare $I^{**}(y_{t_{0}, \varepsilon})$ with $I^{**}(y)$. To this aim, recalling (2.8), we see it is enough to compare
\[ \int_{K_{t_{0}, \varepsilon}^{\pm}} f^{**} \left( y_{t_{0}, \varepsilon}^{\pm}(t) \right) \left( y_{t_{0}, \varepsilon}^{\pm}(t) \right)^{'} \right) dt \text{ and } \int_{K_{t_{0}, \varepsilon}^{\pm}} f^{**}(y(t), y'(t)) dt. \]
As \((y_{t_{0}, \varepsilon}(t), (y_{t_{0}, \varepsilon}(t))^{'}(t))\) can only stay on the upper and lower parts of the boundary of \(D'\) for a.e. \(t \in K_{t_{0}, \varepsilon}^{\pm}\) by (2.11) and (2.12), (3.6) shows that the first integral turns in
\[ \int_{K_{t_{0}, \varepsilon}^{\pm}} f^{**} \left( y_{t_{0}, \varepsilon}(t) \right) \left( y_{t_{0}, \varepsilon}(t) \right)^{'}(t) \right) dt = \int_{K_{t_{0}, \varepsilon}^{\pm}} \left[ d \left( y_{t_{0}, \varepsilon}(t) \right) \left( y_{t_{0}, \varepsilon}(t) \right)^{'}(t) + q \left( y_{t_{0}, \varepsilon}(t) \right) \right] dt. \]
By the fundamental theorem of calculus, the integrals of \(d \left( y_{t_{0}, \varepsilon}(t) \right) \left( y_{t_{0}, \varepsilon}(t) \right)^{'}(t)\) and \(d(y)t\) over the interval \(K_{t_{0}, \varepsilon}^{\pm}\) are equal as they are both derivatives of absolutely continuous functions having the same values at the endpoints of the interval \(K_{t_{0}, \varepsilon}^{\pm}\). Hence, the previous computation together with (3.9) and (3.7) yields that
\[ \int_{K_{t_{0}, \varepsilon}^{\pm}} f^{**} \left( y_{t_{0}, \varepsilon}(t) \right) \left( y_{t_{0}, \varepsilon}(t) \right)^{'}(t) \right) dt \leq \int_{K_{t_{0}, \varepsilon}^{\pm}} f^{**}(y(t), y'(t)) dt + \int_{K_{t_{0}, \varepsilon}^{\pm}} \left[ q \left( y_{t_{0}, \varepsilon}(t) \right) - q(y(t)) \right] dt. \]
Then, recall that \(q\) is monotone on the interval \([\eta_{0} - \delta, \eta \eta + \delta]\) and that (2.9+) and (2.9−) hold. Therefore, setting \(y_{t_{0}, \varepsilon} = y_{t_{0}, \varepsilon}\) and \(K_{t_{0}, \varepsilon} = K_{t_{0}, \varepsilon}^{*}\) if \(q\) is increasing and \(y_{t_{0}, \varepsilon} = y_{t_{0}, \varepsilon}^{-}\) and \(K_{t_{0}, \varepsilon} = K_{t_{0}, \varepsilon}^{+}\) otherwise, we conclude that all functions \(y_{t_{0}, \varepsilon}\) are solutions to \((P^{**})\). Moreover, (3.3a) and (3.3b) follow immediately either from (2.9−) or from (2.9+) and from (2.12) and (3.8) respectively. This completes the proof of the step.

**Proof of Step 2.** We assume again that the set \(E\) defined by (3.2) has positive measure otherwise the thesis of the step trivially holds with \(x = y\).

Then, relying on the construction of the previous step, we associate with every point \(s \in E\) a family of nondegenerate, compact intervals \(\{K_{s, \varepsilon}: 0 < \varepsilon \leq \varepsilon_{0}(s)\}\) contained in \((0, T)\) and a family of new solutions \(\{y_{s, \varepsilon}: 0 < \varepsilon \leq \varepsilon_{0}(s)\}\) to the relaxed problem \((P^{**})\) such that (3.3) hold. Moreover, we can assume that \(0 < \varepsilon_{0}(s) \leq 1\) for every \(s \in E\).

Now, we are left to prove that we can select and glue some of these functions \(y_{t_{0}, \varepsilon}\) so as to find a new solution \(x\) to \((P^{**})\) satisfying (3.5).

To this purpose, recall that the intervals \(\{K_{s, \varepsilon}: 0 < \varepsilon \leq \varepsilon_{0}(s)\}\) defined in the previous step constitute a Vitali covering of \(E\) because of (2.7). Hence, Vitali’s covering theorem yields (at most) countably many points \(s_{h} \in E\) and numbers \(\varepsilon_{h} \in (0, \varepsilon_{0}(s_{h}))\) such that the corresponding intervals \(K_{h} = K_{s_{h}, \varepsilon_{h}}\) are pairwise disjoint subsets of \((0, T)\) which cover \(E\) up to a null set. Let also \(y_{h} = y_{s_{h}, \varepsilon_{h}}\) be the corresponding solution to \((P^{**})\) so that the equality
\[ \int_{K_{h}} f^{**} (y_{h}(t), y_{h}'(t)) dt = \int_{K_{h}} f^{**} (y(t), y'(t)) dt \]
follows from (2.8) and moreover
\[ f^{**} (y_{h}(t), y_{h}'(t)) = f (y_{h}(t), y_{h}'(t)) \quad \text{for a.e.} \quad t \in K_{h} \]
by (3.3b), i.e. the vectors \((y_{h}(t), y_{h}'(t))\) keep off the set \(D\) for a.e. \(t \in K_{h}\).

Then, we set
\[ x(t) = y(t) + \sum_{h} [y_{h}(t) - y(t)], \quad t \in [0, T], \]
and, as in Claim 1 of Proposition 2.3, we show that the series above converges strongly in \(AC([0, T])\). Indeed, the functions \(y_{h} - y\) are absolutely continuous functions on \([0, T]\) whose supports \(K_{h}\) are
pairwise disjoint so that the series above actually reduces to a finite sum for every $t$ and its partial sums are bounded by 1 by either (2.9+) or (2.9−) and the choice of $\varepsilon_0(s)$. Thus, the series converges strongly in $L^1([0, T])$ by Lebesgue’s dominated convergence theorem. As to the derivatives, first recall that the basic assumptions (H2) and (H3) imply that (1.10) holds, i.e. that $f^{**}(\eta, \xi)$ has at least linear growth at infinity as $|\xi| \to \infty$, uniformly with respect to $\eta$ ranging in a bounded interval. Therefore, setting $R = \|y\|_\infty + 1$ for instance and letting $\alpha > 0$ and $\beta \geq 0$ be the corresponding numbers as in (1.10), we get from (3.11) that

$$\sum_h \int_{K_h} |y'_h(t)| \, dt \leq \frac{1}{\alpha} [I^{**}(y) + \beta T].$$

Hence,

$$\sum_h \int_0^T |y'_h(t) - y'(t)| \, dt = \sum_h \int_{K_h} |y'_h(t) - y'(t)| \, dt \leq \sum_h \int_{K_h} |y'_h(t)| \, dt + \|y'\|_1 \leq \frac{1}{\alpha} [I^{**}(y) + \beta T] + \|y'\|_1 < \infty,$$

i.e. the series of the derivatives converges strongly in $L^1([0, T])$ and this proves the claim about the series defining $x$.

Finally, it is plain that $x$ is feasible for $(P^{**})$ because of (2.8) so that, adding (3.11) up for every $h$, we conclude that $x$ is a solution to $(P^{**})$. Moreover, $x = y$ on $[0, T] \setminus (\cup_h K_h)$ whereas $x = y_h$ on $K_h$ and $x' = y'_h$ almost everywhere on the same set so that the equality $f^{**}(x, x') = f(x, x')$ almost everywhere on $\cup_h K_h$ follows from (3.12). As the intervals $K_h$ cover $E$ up to a null set, we conclude that (3.5) holds.

**Proof of Step 3.** Let $x$ be the solution to $(P^{**})$ satisfying (3.5) that was defined in the previous Step 2 and let $A$ be the set defined by (3.4). We have to show that $A$ is negligible and this will be accomplished by showing that, otherwise, a feasible function $\overline{x}$ such that $I^{**}(\overline{x}) < I^{**}(x)$ would exist.

Indeed, recalling the definitions of $A$ and $M$, the set $A$ itself can be actually written, up to a null set, as countable union of sets

$$B_i = \{ t \in (0, T) : x(t) = m_i, x \text{ is differentiable at } t \text{ and } x'(t) = 0 \}$$

since $x'$ vanishes almost everywhere on each level set $\{x = m_i\}$. Now, assume by contradiction that some set $B_i$ has positive measure and, to simplify the notations, set $m = m_i, J = J_i$ and $B = B_i$. Note also that (e) of Proposition 1.1 and our assumption (1.14) imply that $0 \in J$ and that $m$ has to be a strict, local maximum point of $\eta \in D^0 \to Ef^{**}(\eta, 0)$.

Then, for a sufficiently small $\delta > 0$, there exist two functions $a, b : [m - 2\delta, m + 2\delta] \to \mathbb{R}$ as in Proposition 2.1 such that the functions $d, q : [m - 2\delta, m + 2\delta] \to \mathbb{R}$ defined by

$$\begin{cases} 
  d(\eta) = d(\eta, \xi) \\
  q(\eta) = Ef^{**}(\eta, \xi)
\end{cases} \quad \eta \in [m - 2\delta, m + 2\delta]$$

are well defined because of Proposition 1.1, no matter what $\xi \in (a(\eta), b(\eta))$ is, and moreover the following properties hold:

(3.13) $(m, 0) \in D$;

(3.14) $a(\eta) < 0 < b(\eta)$ for every $\eta \in [m - 2\delta, m + 2\delta]$;

(3.15) $q(\eta) = f^{**}(\eta, 0)$ for every $\eta \in [m - 2\delta, m + 2\delta]$;

(3.16) $f^{**}(\eta, \xi) = d(\eta)\xi + q(\eta)$ for every $\xi \in (a(\eta), b(\eta))$ and $\eta \in [m - 2\delta, m + 2\delta]$;
According to this assumption, we choose the + functions and, to simplify the notations, we set
\[ q_{\varepsilon} = q(\varepsilon) = \max\{a(\eta): |\eta - m| \leq \delta\} \] for every \( \varepsilon > 0 \) and assume \( \varepsilon_0 = \varepsilon_0(s, \delta) \) is small enough so that \( \varepsilon_0 \leq 2\delta \) and \( |x(t) - m| \leq \delta \) for every \( t \) in \( J_{s,2\varepsilon_0} \). Hence, \( |x_{s,\varepsilon}^\pm(t) - m| \leq 2\delta \) for every \( t \) in \( H_{s,\varepsilon}^\pm \) and every \( 0 < \varepsilon \leq \varepsilon_0 \) by either (2.4+) or (2.4−). Each function \( x_{s,\varepsilon}^\pm \) is feasible for \( (P^*) \) because of (2.2) and (2.3) and, in order to compare
\[ I^*(x_{s,\varepsilon}^\pm) \text{ with } I^*(x), \] it is enough to compare
\[ \int_{H_{s,\varepsilon}^\pm} f^{**}(x_{s,\varepsilon}^\pm(t), x'_{s,\varepsilon}^\pm(t)) dt \] and
\[ \int_{H_{s,\varepsilon}^\pm} f^{**}(x(t), x'(t)) dt. \]

Now, the very same computations of Step 1 yield that
\[ \int_{H_{s,\varepsilon}^\pm} f^{**}(x_{s,\varepsilon}^\pm(t), x'_{s,\varepsilon}^\pm(t)) dt \leq \int_{H_{s,\varepsilon}^\pm} f^{**}(x(t), x'(t)) dt + \int_{H_{s,\varepsilon}^\pm} [q(x_{s,\varepsilon}^\pm(t)) - q(x(t))] dt \]
for every \( 0 < \varepsilon \leq \varepsilon_0 \) and we claim that, for small enough \( \varepsilon \), we can choose either + or − so that the last summand at the right hand side of (3.19) is negative, thus getting a contradiction.

To see this, choose a decreasing sequence \( \{\varepsilon_k\}_k \) in \( (0, \varepsilon_0) \) such that \( \varepsilon_k \to 0 \) and set
\[ \eta_k = \frac{1}{\varepsilon_k} \sup \{|x(t) - m|: |t - s| < 2p\varepsilon_k\} \]
for every \( k \)
where \( p = \max\{\min\{(\beta - x'(s))^{-1}, (x'(s) - \alpha)^{-1}\}\} \). Obviously, \( \eta_k \to 0_+ \) since \( x \) is differentiable at \( s \) with \( x'(s) = 0 \) by assumption and, moreover, \( 0 < \eta_k \varepsilon_k \leq \delta \) by the choice of \( \varepsilon_0 \). Then, recalling that \( m \) is a strict, local maximum point of \( q \) and possibly extracting a subsequence that we still label as \( \{\varepsilon_k\}_k \), we can assume that the minimum between \( q(m - \eta_k \varepsilon_k) \) and \( q(m + \eta_k \varepsilon_k) \) is actually achieved for every \( k \) by terms with the same sign inside, say \( q(m + \eta_k \varepsilon_k) \), so that
\[ 0 < q(m) - q(m + \eta_k \varepsilon_k) = \max\{q(m) - q(m - \eta_k \varepsilon_k), q(m) - q(m + \eta_k \varepsilon_k)\} \]
holds for every \( k \).

According to this assumption, we choose the + functions and, to simplify the notations, we set \( x_k = x_{s,\varepsilon_k}^\pm \) and \( H_k = H_{s,\varepsilon_k}^\pm \) for every \( k \). Of course, if the minimum between \( q(m - \eta_k \varepsilon_k) \) and \( q(m + \eta_k \varepsilon_k) \) was achieved by \( q(m - \eta_k \varepsilon_k) \) instead, we would have chosen the − functions.

Finally, set also \( J_\varepsilon = J_{s,\varepsilon_0}^\pm \) for \( \varepsilon > 0 \) and note that (2.2) reduces to
\[ J_{\varepsilon_k/2} \subset H_k \subset J_{2\varepsilon_k}. \]

We prove the claim by showing that the integral
\[ \int_{H_k} [q(x_k(t)) - q(x(t))] dt \]
is eventually negative.

To see this, set
\[ A_k^1 = \frac{1}{|H_k|} \int_{H_k} [q(m) - q(x_k(t))] dt \quad \text{and} \quad A_k^2 = \frac{1}{|H_k|} \int_{H_k} [q(m) - q(x(t))] dt \]
for every \( k \) so that the claim reduces to proving that eventually \( |A_k^1 - A_k^2| > 0 \). Indeed, recalling (3.21), that \( q \) is decreasing on the interval \([m, m + 2\delta]\) by (3.18) and noting that (2.5+) reduces to
\[ 2\delta \geq \varepsilon_k \geq x_k(t) - m \geq \varepsilon_k/2, \quad t \in J_{\varepsilon_k/2}, \]
because $s \in B$ and because of the choice of $\varepsilon_0$, we find that
\[
A_k^1 \geq \frac{1}{|J_{2\varepsilon_k}|} \int_{J_{\varepsilon_k}/2} [q(m) - q(x(t))] \, dt \geq \frac{1}{|J_{2\varepsilon_k}|} \int_{J_{\varepsilon_k}/2} [q(m) - q(m + \varepsilon_k/2)] \, dt = \frac{1}{4} [q(m) - q(m + \varepsilon_k/2)]
\]
for every $k$ since $|J_{\varepsilon_k}/2|/|J_{2\varepsilon_k}| = 1/4$ by (2.1). As to $A_k^2$, note that
\[
A_k^2 = \frac{1}{|H_k|} \int_{H_k \setminus B} [q(m) - q(x(t))] \, dt \quad \text{for every } k
\]
and that $m - \eta_k \varepsilon_k \leq x(t) \leq m + \eta_k \varepsilon_k$ for $t \in H_k$ by (3.21) and the very definition of $\eta_k$. Hence,
\[
0 \leq q(m) - q(x(t)) \leq \max \{q(m) - q(m - \eta_k \varepsilon_k), q(m) - q(m + \eta_k \varepsilon_k)\} = q(m) - q(m + \eta_k \varepsilon_k)
\]
for every $t \in H_k$ and every $k$ by (3.18) and (3.20) whence
\[
0 \leq A_k^2 \leq \frac{|H_k \setminus B|}{|H_k|} [q(m) - q(m + \eta_k \varepsilon_k)] \quad \text{for every } k.
\]
Since $\eta_k \to 0$, it follows that eventually $q(m) - q(m + \varepsilon_k/2) \geq q(m) - q(m + \eta_k \varepsilon_k) > 0$ by (3.18). As $s$ is a density point of $B$ and the intervals $\{H_k\}_k$ shrink at $s$, the ratio $|H_k \setminus B|/|H_k|$ goes to zero because of (1.8) and the conclusion follows. \hfill \Box

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PIETRO CELADA, DIPARTIMENTO DI SCIENZE MATEMATICHE – UNIVERSITÀ DEGLI STUDI DI TRIESTE, P.LE EUROPA 1, I-34127 TRIESTE (ITALY)
E-mail address: celada@dsm.univ.trieste.it

STEFANIA PERROTTA, DIPARTIMENTO DI MATEMATICA PURA ED APPLICATA “G. VITALI” – UNIVERSITÀ DEGLI STUDI DI MODENA E REGGIO EMILIA, VIA CAMP1213/B, I-41100 MODENA, ITALY
E-mail address: perrotta@mail.unimo.it