Representation of integers as sums of primes

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The binary Goldbach problem

Goal: give positive lower bound for

\[ r_2(n) = \sum_{p_1+p_2=n} 1 \]

when \( n \to +\infty \) through even values
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The expected value for $r_2(n)$ is

$$\mathcal{S}(n) \frac{n}{\log^2 n} \quad \text{where} \quad \mathcal{S}(n) = 2 \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p|n} \frac{p-1}{p-2}$$
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\( \mathcal{S}(n) \) is an arithmetical correction over the average number of representations
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Heuristic: “double” sieve on the possible solutions of \( n = n_1 + n_2 \)
Heuristic based on the sieve of Eratosthenes

|   | 1 | 3 | 5 | 7 | 9 | 11 | 13 | 15 | 17 | 19 | 21 | 23 | 25 | 27 | 29 | 31 | 33 | 35 | 37 | 39 | 41 | 43 | 45 | 47 | 49 | 51 | 53 | 55 | 57 | 59 |
|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
|+ | + | + | + | + | + | + | + | + | + | + | + | + | + | + | + | + | + | + | + | + | + | + | + | + | + | + | + | + | + | + | + | + | + |
|1 | 3 | 5 | 7 | 9 | 11 | 13 | 15 | 17 | 19 | 21 | 23 | 25 | 27 | 29 | 31 | 33 | 35 | 37 | 39 | 41 | 43 | 45 | 47 | 49 | 51 | 53 | 55 | 57 | 59 |

Removal of 1 class modulo \( p \) if \( p \mid n \), but removal of 2 classes if \( p \nmid n \).

Plausible guess:

\[
R_2(n) \approx n \prod_{p \leq n} \left(1 - \frac{1}{2} - \frac{\omega(p)}{p}\right)
\]

where \( \omega(p) = \begin{cases} 1 & \text{if } p \mid n \\ 2 & \text{if } p \nmid n \end{cases} \)

Right?

A. Zaccagnini (Parma)
Heuristic based on the sieve of Eratosthenes

Plausible guess $r_2(n) \approx n \prod_{p \leq n} \frac{1}{2} - \omega(p)$

where $\omega(p) =
\begin{cases} 1 & \text{if } p | n \\ 2 & \text{if } p \nmid n \end{cases}$
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The binary Goldbach problem

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Remove 1 class mod $p$ if $p \mid n$
Heuristics based on the sieve of Eratosthenes

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Remove 1 class mod $p$ if $p \mid n$, but remove 2 classes if $p \nmid n$

Plausible guess

$$r_2(n) \approx n \prod_{p \leq n^{1/2}} \frac{p - \omega(p)}{p}$$

where

$$\omega(p) = \begin{cases} 
1 & \text{if } p \mid n \\
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This is a well-known phenomenon: compare the PNT with the value of the Mertens product

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Taking this into account and using the Mertens theorem as above, after some tidying up, we find the expected asymptotic formula

$$r_2(n) \sim \mathcal{G}(n) \frac{n}{\log^2 n}$$
Weighted number of representations

Technically easier to work with

\[ R_2(n) = \sum_{m_1 + m_2 = n} \Lambda(m_1) \Lambda(m_2) \]
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Expected asymptotic formula

\[ R_2(n) \sim \mathcal{S}(n)n \]

as \( n \to +\infty \) through even integers
Exceptional set

Let

\[ E(X) = \{ n \leq X : n \text{ is even and } r_2(n) = 0 \} \]
Exceptional set

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Montgomery & Vaughan (1975)

$$|\mathcal{E}(X)| \ll X^{1-\delta} \quad \text{for some } \delta > 0$$
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- compute (do not estimate) the contribution from the possible “exceptional zero” \( \tilde{\beta} \)
- use Gallagher’s PNT (very strong in uniformity, even stronger if the exceptional zero exists)
Exceptional set, II

Pintz (2006)

\[ |\mathcal{E}(X)| \ll X^{2/3} \]
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compute exactly the contribution from “many” zeros of Dirichlet $L$-functions in a thin region just to the left of $\sigma = 1$
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Remark

\[ \frac{x^\rho}{\rho} \]

is large compared to the size $x$ of the expected main term when
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Remark

\[ \frac{x^\rho}{|\rho|} \]

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- $|\rho|$ is small
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is large compared to the size \( x \) of the expected main term when

- \( |\rho| \) is small
- \( 1 - \Re(\rho) \) is small
Standard circle method

Fourier-coefficient formula: for \( n \leq N \)

\[
R_2(N) = \int_0^1 S_N(\alpha)^2 e(-n\alpha) \, d\alpha \quad \text{where} \quad S_N(\alpha) = \sum_{m \leq N} \Lambda(m)e(m\alpha)
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Heuristic: PNT for progressions implies

$$S_N\left(\frac{a}{q}\right) \approx \frac{\mu(q)}{\phi(q)} N + \text{contribution from zeros of } L\text{-functions}$$

uniformly for small $q$ and $(a, q) = 1$
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\[
S_N\left(\frac{a}{q} + \eta\right) \approx \frac{\mu(q)}{\phi(q)} \sum_{m \leq N} e(m\eta)
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Standard circle method

Fourier-coefficient formula: for \( n \leq N \)

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R_2(N) = \int_{0}^{1} S_N(\alpha)^2 e(-n\alpha) \, d\alpha \quad \text{where} \quad S_N(\alpha) = \sum_{m \leq N} \Lambda(m) e(m\alpha)
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Neglect error terms and “large” \( q \)’s: as above, expect

\[
R_2(n) \sim n\mathcal{S}(n)
\]
Goal: Asymptotic formula with “many” terms for

\[ \Sigma_0(N) = \sum_{n \leq N} R_2(n) \]
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Similar to the function

\[ \psi_1(N) = \sum_{n \leq N} \Lambda(n)(N - n) = \int_0^N \psi(t) \, dt \]

that appears in de la Vallée-Poussin’s proof of the PNT
Heuristics

Setting $\psi(x) = x + E(x)$ we find
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$$

$$
= \frac{1}{2}N^2 + \int_0^N E(t) \, dt + \sum_{n \leq N} \Lambda(n)E(N - n)
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Heuristic

Setting $\psi(x) = x + E(x)$ we find

$$\sum_0(N) = \sum_{n \leq N} \Lambda(n)\psi(N-n)$$

$$= \sum_{n \leq N} \Lambda(n)(N-n) + \sum_{n \leq N} \Lambda(n)E(N-n)$$

$$= \int_0^N \psi(t) \, dt + \sum_{n \leq N} \Lambda(n)E(N-n)$$

$$= \frac{1}{2}N^2 + \int_0^N E(t) \, dt + \sum_{n \leq N} \Lambda(n)E(N-n)$$

The explicit formula for $\psi$ implies that $E(x)$ is (essentially) a sum over zeros $\rho$ of the Riemann $\zeta$-function of terms of the form $-x^\rho \rho^{-1}$
Let us compute the total contribution of one such zero $\rho$: by partial summation
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$$- \frac{1}{\rho} \left( \int_0^N t^\rho \, dt + \sum_{n \leq N} \Lambda(n)(N - n)^\rho \right) = - \frac{N^{\rho+1}}{\rho(\rho + 1)} - \int_0^N \psi(t)(N - t)^\rho \, dt$$
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$$= -\frac{N^{\rho + 1}}{\rho(\rho + 1)} - N^{\rho + 1}B(2, \rho) - \int_0^N E(t)(N - t)^\rho \, dt$$

$$= -2 \frac{N^{\rho + 1}}{\rho(\rho + 1)} - \int_0^N E(t)(N - t)^\rho \, dt$$

Formally, this gives the expected “secondary main term” for $\Sigma_0$
Hence, we may write

\[ \Sigma_0(N) = \frac{1}{2} N^2 - 2 \sum_{\rho} \frac{N^{\rho+1}}{\rho(\rho + 1)} + E_0(N) \]

where \( E_0(N) \) is expected to be small
Conditional results

Under RH

- Fujii (1991)

\[ E_0(N) \ll N^{4/3} (\log N)^2 \]
Conditional results

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- Bhowmik & Schlage-Puchta (2010)
  \[ E_0(N) \ll N(\log N)^5 \]
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Limit of the method (without further hypotheses)
A variant of the circle method

Key ingredient

\[
\max_{x \in [2,N]} \left| \sum_{n \leq y} \left( R_2(n) + n - 2\psi(n) \right) e^{-n/N} \right| \ll N(\log N)^3
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A variant of the circle method

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Use circle method in the original setting of Hardy and Littlewood (1923)

\[
\widetilde{S}_N(\alpha) = \sum_{m \geq 1} \Lambda(m)e^{-n/N}e(m\alpha) = \frac{1}{z} + \widetilde{E}(\alpha)
\]

where \( z = N^{-1} - 2\pi i\alpha \), and \( \widetilde{E} \) is “small”
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where \(z = N^{-1} - 2\pi i\alpha\), and \(\tilde{E}\) is “small”

We also set

\[
T(\alpha) = T_y(\alpha) = \sum_{n \leq y} e(n\alpha)
\]
Then

\[ \sum_{n \leq y} e^{-n/N} R(n) = \sum_{n \leq y} \int_{-1/2}^{1/2} \tilde{S}(\alpha)^2 e(-n\alpha) \, d\alpha \]
Then

\[
\sum_{n \leq y} e^{-n/N} R(n) = \sum_{n \leq y} \int_{-1/2}^{1/2} \tilde{S}(\alpha)^2 e(-n\alpha) \, d\alpha \\
= \int_{-1/2}^{1/2} \tilde{S}(\alpha)^2 T(-\alpha) \, d\alpha
\]
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= \int_{-1/2}^{1/2} \tilde{S}(\alpha)^2 T(-\alpha) \, d\alpha
\]

\[
= \int_{-1/2}^{1/2} \frac{T(-\alpha)}{z^2} \, d\alpha + 2 \int_{-1/2}^{1/2} \frac{T(-\alpha)\tilde{E}(\alpha)}{z} \, d\alpha
\]

\[
+ \int_{-1/2}^{1/2} T(-\alpha)\tilde{E}(\alpha)^2 \, d\alpha
\]
Then

\[
\sum_{n \leq y} e^{-n/N} R(n) = \sum_{n \leq y} \int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{S}(\alpha)^2 e(-n\alpha) \, d\alpha
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The first summand gives rise to the main term, the second to the “secondary main term,” since \(\tilde{E}\) is a sum over zeros like \(E(x) = \psi(x) - x\)
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+ \int_{-1/2}^{1/2} T(-\alpha)\tilde{E}(\alpha)^2 \, d\alpha
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Technical difficulties due to the use of infinite series
Then

\[
\sum_{n \leq y} e^{-n/N} R(n) = \sum_{n \leq y} \int_{-1/2}^{1/2} \tilde{S}(\alpha)^2 e(-n\alpha) \, d\alpha \\
= \int_{-1/2}^{1/2} \tilde{S}(\alpha)^2 T(-\alpha) \, d\alpha \\
= \int_{-1/2}^{1/2} \frac{T(-\alpha)}{z^2} \, d\alpha + 2 \int_{-1/2}^{1/2} \frac{T(-\alpha)\tilde{E}(\alpha)}{z} \, d\alpha \\
+ \int_{-1/2}^{1/2} T(-\alpha)\tilde{E}(\alpha)^2 \, d\alpha
\]

The first summand gives rise to the main term, the second to the “secondary main term,” since \(\tilde{E}\) is a sum over zeros like \(E(x) = \psi(x) - x\).

Technical difficulties due to the use of infinite series

For future reference, the relevant range for \(\alpha\) is \([-\frac{1}{2}, \frac{1}{2}]\).
Then
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Yields ET of size \( \ll NL^3 \) (using RH)
Applications

Same technique: improvements on ETs for asymptotics for

$$\sum_{m_1+\cdots+m_k=n} \Lambda(m_1) \cdots \Lambda(m_k)$$

when $k \geq 3$ (using GRH) and $n \equiv k \mod 2$
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See Friedlander & Goldston (1997), AL & AZ (2012)
Cesàro averages of $R_2(n)$

Goal: many-term asymptotic formula for

$$\sum_k(N) = \sum_{n \leq N} \left(1 - \frac{n}{N}\right)^k R_2(n)$$

for $k \geq 0$
Cesàro averages of $R_2(n)$

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for $k \geq 0$. Success for $k > 1$. 

A. Zaccagnini (Parma)
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$$\frac{\Sigma_k(N)}{\Gamma(k+1)} = \frac{N^2}{\Gamma(k+3)} - 2\sum_{\rho} \frac{\Gamma(\rho)}{\Gamma(\rho + k + 2)} N^{\rho + 1}$$

$$+ \sum_{\rho_1} \sum_{\rho_2} \frac{\Gamma(\rho_1)\Gamma(\rho_2)}{\Gamma(\rho_1 + \rho_2 + k + 1)} N^{\rho_1 + \rho_2} + E_k(N)$$
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Then (without hypothesis)

$$E_k(N) \ll_k N^{1/2}$$

for $k > 1$
Cesàro averages of $R_2(n)$

Goal: many-term asymptotic formula for

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Then (without hypothesis)

$$E_k(N) \ll k N^{1/2}$$

for $k > 1$. Probably true for $k > 1/2$
Notice that smoothing does not remove oscillating terms
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Technique: Laplace transforms

\[ \frac{1}{2\pi i} \int_{(a)} v^{-s} e^v \, dv = \frac{1}{\Gamma(s)} \]

where \( \Re(s) > 0 \) and \( a > 0 \)
Notice that smoothing does not remove oscillating terms

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\[ \frac{N^k}{\Gamma(k+1)} \sum_k(N) = \sum_{n \leq N} R_2(n) \frac{(N-n)^k}{\Gamma(k+1)} = \frac{1}{2\pi i} \int_{(a)} e^{Nz} z^{-k-1} \tilde{S}(z)^2 \, dz \]
Cesàro averages

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\]

Use

\[
\tilde{S}(z) = \sum_{m \geq 1} \Lambda(m) e^{-mz} = \frac{1}{z} - \sum_{\rho} z^{-\rho} \Gamma(\rho) + E(y, a)
\]

where \( E \) is small
Square out $\tilde{S}$ and integrate, using Laplace transform
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Technical difficulties
Square out $\tilde{S}$ and integrate, using Laplace transform

Technical difficulties

- Exchange series and integrals
Square out $\tilde{S}$ and integrate, using Laplace transform

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- Convergence of double sum over zeros
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Thank you